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ABSTRACT

This document contains 15 research papers presented at the Fourth and Fifth Annual Conferences of the Research Council for Diagnostic and Prescriptive Mathematics. The papers are organized into four sections. In part I, four position papers by Romberg, Engelhardt, Ashlock, and Heddens reflect research ideas which underlie the Council. Part II contains papers on diagnostic testing by Capps and Jenkins, Sadowski, Denmark and Mumoz, and Johnson. Analysis of types of errors students make is the focus in part III, with papers by Grady and Hutcherson, Gregory, Bright, and Dockweiler. Part IV includes papers by Wiebe, Scheer, and Hutchings on aspects of prescriptive instruction. (MNS)

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MONOGRAPH

RESEARCH REPORTS FROM THE RESEARCH COUNCIL ON DIAGNOSTIC AND PRESCRIPTIVE MATHEMATICS

Edited by

Thomas A. Romberg
University of Wisconsin

RESEARCH REPORTS
FROM

THE FOURTH AND FIFTH NATIONAL CONFERENCES
ON DIAGNOSTIC AND PRESCRIPTIVE MATHEMATICS
(Revised Third Edition)

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PRESCRIPT

Chapter 1

INTRODUCTION TO THIS MONOGRAPH

Thomas A. Romberg
University of Wisconsin

The purpose of preparing and publishing this monograph by the Research Council on Diagnostic and Prescriptive Mathematics (RCDPM) is to make available to members of the Council a set of papers presented at the fourth and fifth national conferences on Diagnostic and Prescriptive Mathematics. These conferences are the initial two conferences sponsored by the newly formed Research Council. As the Vice-President for Research for RCDPM, it has been my task to select, organize and edit a representative set of papers from those conferences which reflect both the thoughts about research by the founders and leaders of the Council, and the kind of research that members are doing. Fifteen papers are included in this monograph. Before describing these papers and how they are organized, let me briefly describe the origins of RCDPM and its status as a professional organization in mathematics education.

In 1974, Professor James Heddens of Kent State University organized the "First National Conference on Remedial Mathematics," which was subsequently held on the Kent State Campus. Two aspects of the title of the conference should be noted: It was called the "First" conference. Clearly continued dialogue via later conferences was envisioned. Additionally, remediation was emphasized. Most of the participants of that conference train persons to be mathematical clinicians. While schools often have trained personnel to work with students who have reading difficulties, few have similarly trained staff members to work with students with mathematical difficulties. To a large extent, the intent of the first conference was to see if there was interest and common concerns among the participants involved with mathematical clinics to continue the dialogue. By the Third National Conference on Remedial Mathematics held at Kent State University in 1976, it became apparent an organization needed to be formed. Under the guidance of a steering committee, a constitution and by-laws were developed, approved and implemented. The title chosen for the new organization was Research Council for Diagnostic and Prescriptive Mathematics. At the Fourth National Conference, held at the University of Maryland, the new organization became a reality. Officers were elected and over 200 persons joined.

After two full years of life, the RCDPM is gaining momentum and becoming a strong professional organization. However, as with any infant organization, there are many plans and decisions to be made and some problems to be faced. We need to organize and foster research in this area. Diagnostic instruments, techniques of diagnosing and unique methods of corrective teaching need to be developed and studied.

The papers in this monograph have been organized into four sections which reflect those concerns. In Part I, four position papers are presented. They are authored by founders and officers of the Council. They were chosen because they reflect, in part, the ideas about research which underly the organization. The first paper, which I was asked to write as Vice-President

for Research, was the opening research address at the Fifth National Conference. It reflects my thinking about the research agenda and procedures the organization should be following. The next paper, by Jon Engelhardt, who is the treasurer of RCDPM, is an excellent review of the psychological research on alternate expressions of understanding. In particular, mathematical ideas can be expressed through actions, images and symbols. The third paper in this section, by Robert Ashlock, who is President-Elect, is a review of research on numerals for whole numbers. It includes some diagnostic examples and has some provocative instructional ideas which should be examined. The final paper in this section is by James Heddens, the first President of the Council. In his paper he presents the logical analysis of addition facts for memorization. He argues this is an example of the first step in a four step research procedure which should be followed.

In Part II, four papers on diagnostic testing are presented. In the first paper, Lelon Capps and Charles Jenkins review eighteen diagnostic instruments in mathematics commonly used in studies on remediation. Scholars contemplating using one of these tests would be well-advised to heed their comments. In the second paper, Barbara Sadowski uses probabilistic moding procedures to examine five domains tested by the Maryland Diagnostic Arithmetic Test. Next, Tom Denmark and Rafael Mumoz present a format and some examples of a diagnostic model which can be used by classroom teachers. In the last chapter in this part, Martin Johnson presents eighteen Piagetian tasks which could be used for classroom diagnosis. Selecting and constructing tests is clearly a major focus of RCDPM members.

Part III contains four papers which reflect the interest of most clinicians in analyzing the types of errors students make. In Chapter 10, Tim Grady and Lyndal Hutcherson present an analysis of common errors in computation for third, fifth and seventh grade students. In the next paper, John Gregory presents data from a study designed to identify whether the errors children make are mathematical or clerical. George Bright's paper is a review of three studies carried out by Soviet researchers on the kinds of errors retarded and incapable students make. Clarence Dockweiler's study concludes this part. He did an extensive cross-sectional survey on spatial frames of references which represents a different type of error analyses.

Part IV concludes the monograph with three papers whose focus is on prescriptive instruction. James Wiebe carried out a clinical experiment where the level of verbalization was manipulated during instruction on place value using physical models. Janet Scheer presents data on the effects of a supplemental clinic on pupil achievement, attitude and self-concept. In the last chapter, Bart Hutchings presents an overview of several studies on the use of "low-stress" algorithms.

In summary, I believe this set of papers reflects the state-of-the-art in the specialized field of diagnostic and prescriptive mathematics. I confidently expect that as the organization grows and our skills improve, we will produce more and better strategies for studying children's learning difficulties, better diagnostic instruments, better understandings of how errors are made and what contributes to errors and better diagnostic procedures for remediating difficulties.

PART I

POSITION PAPERS

Chapter 2

"Children with Mathematical Learning Difficulties:" A Potential Research Program for the Research Council on Diagnostic and Prescriptive Mathematics

Thomas A. Romberg
University of Wisconsin

In my role as Vice-President for Research of the Research Council on Diagnostic and Prescriptive Mathematics, I intended to accomplish four things for the membership in the paper:

1. to present a political argument on the need to carry out an extensive research program on children with mathematical learning difficulties.

2. to identify recent works which should give conceptual and methodological direction to the extensive research program that I believe is needed.

3. to reraise the question of what paradigm of research should we be following.

4. to review the research concerns of this Council.

THE POLITICAL ARGUMENT

In general, educators are only now beginning to sense the importance of being politically active. Resources to carry out important educational activities are very limited. The scarce resources will only be allocated to "our" activities through the political lobbying process.

To successfully lobby a specific research program, three questions must be answered:

1. Is there a need for the knowledge and products from such a research program?

2. Is there a reasonable probability that the proposed program will fulfill that need?

3. Among other programs, what priority does this program have?

Need. Demonstrating that a genuine need exists involves gathering evidence that survival, health or some other defensible end is not now being adequately serviced (Scriven, 1974, pp. 8-9).

There is ample evidence that a sizeable proportion of children have mathematical learning difficulties. Legislatures are generally aware of the publicity that has surrounded "declining" SAT scores," "how U.S. children compare unfavorably with those in other countries," "the NAEP population data," "a variety of state assessment results," "Competency Based Testing," etc. With this immense and growing data base, the Council needs to first prepare an initial document which synthesizes this data. This should be a strongly stated position paper on the facts related to the extent of mathematical illiteracy. Second, the Council should prepare a document which identifies the nature of mathematical difficulties, their diagnoses and recommended remedial programs or procedures.

To do this now, we need a better knowledge base about:

a) The nature of the children's learning difficulties. Terms like "math phobia" are important in publicizing the problem but are almost useless to the clinician. A catalogue of specific difficulties needs to be developed.

b) Appropriate methods of identifying difficulties need to be specified.

c) Proven techniques of remediating difficulties should be cutlined.

Probability of Success. Unfortunately, good intentions do not necessarily produce good research programs. No matter how genuine the need, the connection between what is proposed and satisfying the need must be demonstrated.

Satisfying a need starts with a serious attempt to understand the problem. It also means that one recognizes that complex problems--like children's mathematical difficulties--are only understood in stages and that proposed work is best postulated in terms of one's current knowledge of that complex problem. Thus, an idea, no matter how bold or imaginative it may appear, must be rooted in a sound theoretic or philosophic rationale. From this position, the new research program should promise that there will be a fundamental change from current practice and that such a change should have practical, positive consequences that are politically viable. Too many educational researchers and developers have failed to take this problem analysis step seriously. The consequences of this are the "cults," "fads" and "sure fire cures" that inundate today's educational scene.

A final factor at this stage which is sometimes overlooked is staff capability. Good ideas can only be translated into good research by capable, trained persons.

In reality, this "probability of success" argument can only be made by individual researchers when they propose a particular line of research. This is in part due to how research is funded in this country. Projects, not bureaucracies are funded. Thus, the Council can prepare important background documents on mathematical difficulties but should not submit proposals to carry out research. However, the Council should be able to help its members build their arguments.

Priority. It is naive to assume that there are adequate human and financial resources to carry out all needed research no matter how genuine or what probability of success. Funding agents have to determine how to allocate their scarce resources. Educators are becoming increasingly aware of the importance of this political decision. Obviously, if a research program on children's mathematical learning difficulties is to be carried out, it must be given high priority by those who allocate resources. However, priority designation by itself will not produce good research. Only by combining need with success probability and priority designation can a research program reasonably proceed.

The basic strategy to get a high priority status usually follows one of two paths. First, if what is being proposed is in line with funding agent's established priorities, proposals should highlight how they relate to those important goals. Second, if what is being proposed is not in line with the established priorities, then a more elaborate argument must be developed. One good way has been suggested by Grotelueschen and Gooler (1972). They indicate that since goals are preferenced-based, one can develop a strong argument by gathering preference data as to why a particular constituency would prefer this proposed line of research even though it does not fit in existing priority designations.

I believe the research program the Council members should be carrying out is in line with current Federal priorities. In fact, last year, Secretary of Health, Education and Welfare, Joseph Califano, outlined several steps the administration "can and should take" to study the reasons why students fail to do well on tests. Specifically, Califano said:

> We need to know who is not performing well, why they are not performing well, how this failure affects students' lives and what we can do to improve their basic skills. The knowledge we gain will permit states and localities to design effective remedial programs as an integral part of their testing effort (Education Daily, November 10, 1977, p. 6).

Mathematics is certainly one area of basic skills where many students have difficulty.

The Council's immediate problem is to convince the Federal bureaucrats that its members have the capability and experience to work on these problems. Thus, I propose that the Council should prepare still another document outlining the strengths of its membership in relationship to the study of Children with Mathematical Learning Difficulties.

In summary, I believe the members of the Research Council can build a sound political argument for establishing a research program on children with mathematical learning difficulties. To accomplish this, I make the following recommendations:

Recommendation #1

The Research Council for Diagnostic and Prescriptive Mathematics appoint a Task Force to prepare three position documents on: 1) the extent of mathematical illiteracy; 2) the nature of children's mathematical difficulties; and 3) the capability of the RCDPM membership to study these problems. These position documents could then be used by the Council's membership in submitting proposals.

Recommendation #2

The Research Council for Diagnostic and Prescriptive Mathematics appoint a Standing Review Panel. The purpose of this review panel would be to professionally critique members' research proposals. This type of assistance at an early stage in proposal preparation should produce both better proposals and more funding.

RECENT WORKS

In this second part, I intended to identify and examine a few of the most important recent works that might give direction to the task force in preparing the document which describes the nature of children's mathematical difficulties. I planned to build a matrix with the column headings being major areas of research such as, identifying and analyzing errors, identification of cognitive skills, specific learning difficulties, pedagogical procedures related to learning difficulties, etc. and the row headings being conceptual papers, methodological papers and empirical studies. Unfortunately, this task proved to be unmanageable without more structure. Very few studies fit nicely into this matrix. As I wrestled with this problem, what evolved is a framework which could be used as a starting point. This framework is based on a stimulus-cognitive processing-response orientation with three external sources which may cause responses to be judged inadequate. Then, I identified one or more recent works which I believe reflect seven aspects of the frameworks.

To describe the framework, let me start with the term "difficulty." A child is judged to have a "learning difficulty" when his responses to a set of stimulus questions is judged inadequate based on some apriori criteria for adequacy. There are four questions about this judgment procedure which need to be raised: 1) What are the specifications for the choice of stimulus questions?; 2) How are responses coded?; 3) What is the criteria for judging adequacy of responses?; and 4) If responses are judged inadequate (the child has a difficulty), what might be the cause? I raise all four of these questions because too many researchers are only interested in the last question and fail to consider the implications of the others in their work.

Stimulus Characteristics. For the preparation of RCDPM position papers, all of the stimulus situations should involve mathematical tasks. But, the

choice of stimuli is an ontological problem in that we often only consider a small portion of mathematical tasks. What I am arguing is that since our ability to analyze responses is dependent upon the choice of stimuli, we should expand our ideas about the stimuli to be used. In particular, the usual approach using behavioral objectives is insufficient in that by concentrating on responses, we often restrict stimulus choices.

The following four papers reflect the kind of content analysis I believe is necessary.

Kieren, T. E. "On the mathematical, cognitive and instructional foundations of rational numbers." (1975)

In this paper, Kieren identified seven different interpretations of rational numbers (as fractions, equivalence classes of fractions, ratio numbers, operators, elements of a quotient field, measures and decimal fractions).

Moser, J. M. "Young childrens representation of addition and subtraction problems." (1979)

Moser carries out an extensive analysis of a variety of problem situations which can be represented with addition and subtraction sentences.

Heddens, J. W. "A theoretical study of the organization of basic addition facts for memorization." (Chapter 5 of this monograph).

In his initial logical analyses of basic addition facts, Heddens is able to classify them into several structural categories.

Erlwanger, S. H. "Benny's conception of rules and answers in IPI mathematics." (1973)

The importance of expanding stimuli is clearly demonstrated in this case study. When Erlwanger presented Benny with mathematical problem types which were different from those in the mathematics program, an illuminating picture of how Benny conceived mathematics was revealed.

Methods of Coding Responses. The standard (1, 0) coding of responses (1 for correct, 0 for incorrect) and then counting the number of correct and incorrect responses is an inadequate conceptualization of the response repertoire for any individual. On this topic, I have chosen three references which reflect the direction of work in this area.

Cronbach, L. J., et al. The Dependability of Behavioral Measurements. (1972)

In this seminal book, Cronbach and his associates present an extensive multifaceted observational approach for behavioral measurements.

Sternberg, R. J. Intelligence, Information, Processing and Analogical Reasoning: The Componential Analysis of Human Abilities. (1977)

Sternberg's work, based on Cronbach's approach, should be read as a guide to how behaviors involve an intensive task analysis of the stimulus, developing a mathematical model for anticipated response characteristics and then classifying responses in terms of that particular model. A second aspect of coding is going beyond judging just-written responses or selected verbal responses. Two papers which reflect this concern are:

Kilpatrick, J. Analyzing the solution of word problems in mathematics: An exploratory study. (1967)

Small, M., Romberg, T., and Carnahan, R. Teacher-student observation procedures and the training of observers for coordinated study #1. (1979)

Given that responses have been coded, cluster analysis is a powerful procedure for classifying either individuals or items.

Hubert, L. and Baker, F. "Data analyses by single-link and complete-link hierarchical clustering." (1976)

Finally, if we are going to get at anything alike second level diagnosis, we are going to have to use tailored or sequential testing procedures. To assume that one can do a complete diagnosis from a single test administration is foolish. Sequential testing is essential if we are going to be able to actually diagnosis a student's performance. The main reference in this area is:

Cronbach, L. and Gleser, G. Psychological Tests and Personnel Decisions. (1965)

Criteria for Adequacy. For many responses, it becomes quite difficult to judge its adequacy. There are two aspects of the problem. The first is that any response reflects something about how the child processes stimulus information. Thus, all responses should be coded in terms of implied cognitive processing. Judgment of adequacy then is a secondary coding. The second concern is on what basis does one judge adequacy? For most situations, the procedure is to use frequency of specific responses to a sample of stimulus situations as the basis upon which judgments can be made. Then some criteria such as 80% of the responses being similar is chosen.

The choices of percentage of similar responses and the number of stimulus situations to be used are rarely addressed in the literature. Two papers I am familiar with that address these questions are:

Kriewall, T. "Applications of information theory and acceptance sampling principles to the management of mathematics instruction." (1969)

Braswell, J. "The formative evaluation of patterns in arithmetic-grade 6, using item sampling." (1970)

Plausible Causes of Inadequate Responses. Given that we can code responses and judge their adequacy, the next problem is to identify plausible causes. Inadequate responses, I believe are likely to be caused by one of four underlying deficiencies. They are: cognitive processing deficiency, poor instruction, lack of motivation and other social-cultural factors.

What if for the problem 35 minus 18, the child responds 23. We can judge that the response is incorrect but where is the difficulty? Is it lack of memory load?; is it poor instruction?; or what? To design a potential remediation program, we need to go beyond identification as an error pattern. Carpenter's review of cognitive processing in mathematics indicates that there are two underlying categories of processes which should be looked at: relational understanding (seriation, conservation, class inclusion, etc.), and information processing capability (in particular, memory load).

Carpenter, T. "Cognitive development research and mathematics education." (1979)

What we need to do is guess where the processing of information is being done incorrectly. I talked about this in more detail in the paper I gave at the Maryland RCDPM conference.

Romberg, T. A. "Second-level diagnoses: Assessing underlying processes not just objectives." (1977)

If diagnosis is to be meaningful, then we should be willing to guess why an individual is making the kind of responses we have coded. What we need is a catalog of specific difficulties derived from lots of studies. I am going to be critical, but we seem to be more interested in remediation than in understanding plausible causes. It is somewhat natural, since we, like doctors, want to treat patients in trouble.

A plausible reason for inadequate responses is poor instruction. In particular, it seems reasonable that if students are not given an opportunity to learn, they are not going to perform well. This has become particularly important in the recent "Time-On-Task" studies. The most important of these studies is the Beginning Teacher Evaluation Study (BTES).

Fischer, C., et al. "Teaching and learning in elementary schools: A summary of the beginning teacher evaluation study." (1978)

The next plausible cause for inadequate performance deals with inadequate motive, desire or readiness. An individual's willingness to work on a task (and a belief he can do it) is as important as his actual capability. The old axiom "you can lead a horse to water, but can't make him drink" is very true in educational settings. Part of what we need to do is figure out ways in which we can get students to drink of the knowledge that we have to offer. A useful reference on this topic is:

Ghatala, E. S. "Effectiveness of individually guided motivation: A summary of the empirical evidence." (1975)

The social culture in which schools operate may be a cause of inadequate performance since it impinges upon both of the selection of stimulus problems and the expected responses. Researchers studying the school culture of the classroom have identified the curriculum of the home, and the hidden curriculum of the school as variables which strongly influence the kinds of problem situations and responses that are allowed. We frankly do not know enough about such influences to be able to discuss their relationship to mathematics. However, it is obvious that the culture within an urban school is quite different than that considered by most educators. A good reference on this is:

Apple, M. Ideology and Curriculum. (1979)

In summary, my attempt to review studies led to the development of a framework from which one could begin to examine studies. It is clear to me that I and probably most of us, have been far too parochial in our interests in mathematics education. What I am arguing is that we need to go considerably beyond our present work, if we are going to develop a reasonable catalog of learning difficulties.

ALTERNATE PARADIGMS FOR RESEARCH

The point I want to emphasize can be argued as follows. Most math educators hold a view of science which is extremely narrow. We unthinkingly appropriate a reconstructed logic of science. We have assumed that a unitary scientific method exists and the surest guarantee of success is to follow the procedures in science as closely as possible. As early as 1929, Dewey noted these tendencies:

> 'educators have' regarded science as a guarantee that goes with the goods rather than a light to the eyes or lamp to the feet. It is prized for its prestige value rather than as an organ of personal alumination of liberation (p. 92).

This point of view needs to be replaced with a more dynamic approach to inquiry. Lee Shulman, in his recent AERA paper, argues for a particular frame of reference based on the work of Robert Merten's (1978). Research on mathematics teaching has been obsessed with design, the study of planned or engineered continuity. Although there exists a tradition of error and analysis in the study of mathematics education, it has generally been focused on the specification of remediation strategies rather than the understanding of the underlying cognitive processes which account for those errors.

Shulman's paper is important because it reflects the crisis surrounding all educational research. Education is in a critical period with respect to growth of knowledge. I happen to like Kuhn's "structure of scientific

revolutions" as a way of explaining what is going on (1962). His model combines periods of slow growth "normal science" with periods of crisis of revolution. Normal science is guided by a paradigm that defines problems and dictates methods. The attention of scientists is directed towards these problems exclusively. Knowledge grows piece by piece building on previous work. The scientist is in essence a puzzle solver rather than a philosopher. In time, however, phenomenon the dominant paradigm cannot explain become important. Anomalies can no longer be ignored. Then, there is a period of crisis when the old is under attack and a new is being sought. True, "new paradigms" are often resisted by those in power, but gradually, the anomalies and problems become too large and the dominant paradigm is over-thrown. At present, I think we are in such a period. The agricultural-psychometric model which has dominated so much of educational research in the last 25 years is now under serious challenge from several fronts. What is important to us as members of RCDPM is that it is more important to study individual learning difficulties from different perspectives than it is to have adequate sampling, high reliability, statistical significance, etc.

COUNCIL CONCERNS

I think there are three primary problems we should be addressing. The first is the funding of research. Funding is a political decision. The board of directors should become involved in the political funding process. My recommendations in the first section of this paper on preparing documents which reflect the position of the organization on research are important. The second problem is to provide assistance to members. We need a procedure to assist members in proposing and carrying out research. The third, and probably the most important problem is how can we develop collaborative research among our members? Like any organization, we have a disparate group of members. While we are all interested in basically the same problem, children with learning difficulties, our interest stems from different sources. Although we call ourselves a research council, there seems to be more interest on remediation than there is in research on remediation or diagnosis. Clearly, the program of the last three meetings have reflected a disjoint interest in workshops. Sociologists who have studied the problem of collegial relationship among scholars find it very difficult to automatically create collegial relationships among people who have not lived with each other and argued over issues for some period of time. It will be interesting to see if over the next few years collegial relationships can be developed within an organization which has a general focus such as RCDPM.

SUMMARY

In this overview, I have tried to do four things. First, the political argument for diagnostic and remedial research in the learning of teaching in mathematics must be made. Second, a framework such as that that I suggested should be used in building a catalog of difficulties. Third, I argued the importance of approaching research using different paradigms. We need to be as creative in our approach to research as we can be. Finally, there

are concerns that this Research Council needs to face. I think we can become an important organization. I think we are getting better. Each year I look forward to more exciting programs, better publications and a better conceptualization of the overall problem of both studying and doing something about children's learning difficulties.

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Chapter 3

FROM PHYSICAL ACTIVITY TO ABSTRACT SYMBOLS: LEVELS OF MATURITY IN DIAGNOSIS AND INSTRUCTION

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In helping teachers understand children as learners of mathematics, I often ask them to briefly interview a child on some particular notion of mathematics. During one such interview, a teacher asked a boy named Bill what a triangle was. Bill's immediate response was to pick up his pencil and draw a rough sketch of a triangle. When questioned further, Bill was unable (or unwilling) to verbalize any information about triangles; he continually responded with another sketch of a triangle. What is Bill's understanding of triangle? Does he really understand? What information has been gained which might aid further instruction on triangles?

Given Bill's situation, some teachers seem to conclude that Bill knows very little about triangles; the fact that Bill did not seem to deal with triangles abstractly was convincing evidence of his lack of understanding. These teachers usually suggest that further instruction on triangles should proceed from the beginning, assuming very little understanding. Other teachers quickly point out that if Bill were a young child, his response might not be so bad. Further instruction, for these teachers, would consist of experiences which build upon the child's present level of functioning. Such teachers seem to have an intuitive sense of "understanding" as a dynamic process where dealing with concepts abstractly is the ultimate step.

It is the thesis of this paper that understanding mathematical notions can be expressed through actions, images and symbols and that these alternate expressions of understanding may be considered levels of maturity.

THEORIES OF LEARNING AND INTELLECTUAL DEVELOPMENT

Since the introduction of Pestalozzian teaching methods, about 1800, some educators and psychologists have been talking about how optimum instruction and learning proceed from concrete to abstract.

One of the most widely known of modern psychologists is Jean Piaget. Piaget has identified four stages of mental growth--stages which inherently stress the concrete-to-abstract nature of learning. According to Piaget, for the first several years of the child's life, mental activity is particularly tied to the active manipulation of objects. At a later age, the child's manipulations begin to be internalized as visual images and ultimately as linguistic, numeric or geometric symbols (Adler, 1966).

The theories of two other modern psychologists have essentially been based upon the views of Jean Piaget. Through his own extensive research, one of these psychologists, Zoltan Dienes (1965, 1967a, 1967b) has evolved four "principles" of mathematics concept learning. Threaded through his explanations of these principles are the notions that, to attain abstract generalization, (1) children should experience concepts in various perceptual settings, including "concrete materials" and "pictures," and (2) concrete settings should precede more abstract settings.

The second psychologist whose theories reflect Piaget's influence is Jerome Bruner. According to Bruner (1960), intellectual development can be characterized in three stages, each distinguishable by the characteristic way in which children view the world and explain it to themselves. Bruner stated that for children, events and objects in the world are at first defined in terms of actions taken toward them. Later, objects and events are somehow picturable or conceivable without action; images are formed as summerizers of action. Ultimately, symbols are used to represent the world, which gradually gain an independence of their own. Describing these stages as systems of representation, Bruner suggested these systems to be mediators through which human thought gains increasing independence from the immediate environment.

Modern psychologists, whose theories are not based upon the ideas of Piaget, have also described learning as stemming from concrete experiences. Galperin (1967), a Soviet psychologist, has postulated learning as the transfer of knowledge from physical action to mental action. Characterizing learning as passing through five stages, Galperin suggested that the action is separated from its physical setting and transferred to audible speech, where its basis is transformed from objective to linguistic verbal. Slavina (1967) extended Galperin's ideas to include imagery as a "necessary" intermediate step.

Other psychologists have looked upon learning as the processing and storing of information. In information processing theories, information is stored and transferred between different types of "stores." These stores are distinguished primarily by their temporal properties; however, other properties have also been specified. In a review of multistores theories and research, Craik and Lockhart (1972) reported the three basic stores to be "sensory stores," "short-term stores" and "long-term stores;" most information processing psychologists have agreed that the three stores can be distinguished by the format of information stored--sensory specific (sensory stores), visual/verbal representations (short-term stores) and semantic/linguistic (long-term stores). Although it may stall in any store, information is thought to flow from sensory to short-term to long-term stores.

Reys (1971) summarized four principles subscribed to by most modern psychologists:

1. Learning is based on experience;

2. Sensory learning is the foundation of all experience and thus, the heart of learning;

3. Learning proceeds from concrete to abstract, and

4. Learning requires active participation by the learner.

All four principles could be identified within each of the theories discussed since all emphasize the importance of concrete experience and the concrete-to-abstract sequential nature of learning. A more detailed analysis of these theories, however, leads to the observation of three other principles.

(1) The concrete-to-abstract sequential nature of learning pertains to concept attainment as well as general intellectual development.

The theories of Dienes and the information processors are concerned with specific content or concepts. However, the application of Piaget's and Bruner's theories to specific content is not at first obvious. In applying his learning theory to formulating a theory of instruction, Bruner (1966) intended his stages of intellectual development to describe stages of concept attainment. For example, he described a teaching procedure for the idea of mathematical group in which the concept was represented concretely through the rotations of a book, graphically through a matrix, and then abstractly through the verbalization of a defining mathematical property.

As for Piaget, referring to the not uncommon example of advanced mathematics students groping for some concrete model of an abstract concept, Reys and Post (1973) suggested Piaget's work implies that in different conceptual areas, learners may be functioning at different stages of mental growth. Thus, Piaget's theory, like Bruner's, appears to have implications (and applications) for concept attainment as well as the more general intellectual development. As Skemp (1971) stated,

> But it may well be the case that we all have to go, perhaps more rapidly than the growing child, through similar stages in each new topic which we encounter--that the mode of thinking available is partly a function of the degree to which the concepts have been developed in the primary system. (p. 66)

(2) Learners can express their understanding of mathematical notions through actions, images and symbols.

With the exception of information processing theory, the theories of the other psychologists have suggested the importance of object manipulation, visual imagery/pictures and symbolization in representing or promoting mental activity. Similarly, information processors have viewed perception, visual/verbal images and abstract symbolism as differentiating types of memory. It seems reasonable that if learners' mental activity is tied to and proceeds from sensory-motor activity to mental images to symbolisms, then learners should similarly express their level of mental processing through manipulative, graphic and symbolic behaviors.

Some of the research on human memory, visual imagery[1] research in particular, appears to support this conclusion. For example, in a study of three-year-olds who had been instructed to remember under which of three cups a toy dog was hiding, Wellman, Ritter and Flavell (1975) noted that the children looked toward and touched the to-be-remembered cup more often than the others. Such overt behaviors rather obviously indicated the level of mental activity for these subjects.

(3) In both general intellectual development and concept attainment, sensory-motor activity probably preceeds visual imagery which in turn probably preceeds symbolization.

Generally speaking, all the theories (and the visual imagery research) suggest that the action-image-symbol sequence of mental activity is the natural order in the development of human understanding. Bruner, the strongest proponent of this developmental order indicated this sequence to be cumulative as well as ordered.

> What is abidingly interesting about the nature of intellectual development is that it seems to run the course of these three systems of representation until the human being is able to command all three. (Bruner, 1966, p. 12)

This suggests that either individuals must have experiences representing thought with actions, images and symbols or, they must be able to reconstruct such experiences from existant cognitive structures.[2]

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[1]It is interesting to note that other research on visual imagery has provided further support for the overt formation of visual images as an intermediate stage in intellectual development. Besides indicating that visual imagery facilitates memory in general (Rowe & Paivio, 1971), research has shown that children above the age of six or seven are able to generate visual images (Levin, Davidson, Wolff & Citron, 1973), while younger children are not (Wolff & Levin, 1972). For these younger children (down to ages three or four) memory is facilitated through relevant motor activity (Wolff & Levin, 1972; Wolff, Levin & Longobardi, 1972; and Levin, McCabe & Bender, 1974). Levin, Ghatala, DeRose, Wilder and Norton (1974) found that by fifth or sixth grade, both pictures and words activated visual images. Similar to the theories of general intellectual development, the facilitation of memory appears to proceed from motor activity to the perception of objects/pictures to words.

[2]In as much as most learning theories appear to describe the evolution of human thought as a gradual disassociation of mental activity from the environment, it seems natural that the ordered action-image-symbol stages of mental activity be conceptualized as "levels of maturity." If this conceptualization is appropriate, then it seems reasonable that a learner can possess concepts without being able to symbolize/verbalize them. Caron (1968) demonstrated this to be true, at least at the visual image level.

The research evidence on the action-image-symbol ordering in concept attainment has not been firmly supportive. For example, while Scott and Neufield (1976) found no significant difference between manipulative and pictorial approaches in effecting concept formation, Ekman (1967) reported a manipulative approach to be better than a pictorial one and both to be better than a symbolic approach; more to the point, Punn (1974) indicated that manipulative-pictorial-symbolic and manipulative-symbolic approaches stimulated higher achievement and attitude than a pictorial-symbolic one. As Scott and Neufeld suggested, the considerable variation in terminology makes research conclusions difficult to interpret.

Challenging the entire notion of order, Wittrock (1973) claimed that regardless of whether actions, images or symbols represent or stimulate learning, the important point is the conscious, active involvement (generation) of the individual in learning. He stated that the "learner must actively construct meaning if he is to learn with understanding (p. 30)." To test his claim, Wittrock conducted a brief study with fifth graders on word definitions; he reported that generating a picture to help remember each definition was effective while tracing a picture or copying a definition were equally ineffective.

Although the action-image-symbol stages for concept attainment and intellectual development appear to be ordered, the lack of confirming research and the research and theorizing of Wittrock suggest that this conclusion may be accepted only tentatively.

IMPLICATIONS FOR DIAGNOSIS AND INSTRUCTION

The thesis of this paper has been that mathematical notions can be expressed through actions, images and symbols, which, in turn, may be conceptualized as levels of maturity. Support for this thesis has been suggested from the psychological theories and base research of Piaget, Dienes, Bruner, Galperin and information processing psychologists. Specifically, it was concluded that (1) besides general intellectual development, specific mathematics understandings (concepts) are attained in a concrete-to-abstract sequence, and (2) the major steps in this sequence-manipulation, visual imagery and symbolization--can be expressed through actions, visual images and symbols. If the steps in attaining understandings are ordered (as they appear to be), then it was suggested that such steps may reflect levels of conceptual maturity.

The implications of this paper's thesis seem to be more commonly applied to developmental and corrective instruction in mathematics than to diagnostic procedures. Indeed, a number of mathematics educators have attempted to apply the action-image-symbol sequence of concept attainment to mathematics instruction (Underhill, 1972; Reisman, 1972; Heddens, 1974; Sowell, 1974).

The major implication is for sequencing educational experiences. If the development of mathematics concepts naturally proceeds from manipulations to visual images to symbolization, then instruction should be such

that it promotes this natural sequence. This is ordinarily achieved through providing a sequence of experiences with manipulative materials, pictures or diagrams, and verbal/numeric symbols. The relative success of the Developing Mathematical Processes (Romberg, et al., 1975-76) mathematics program (Hubbard, 1972) testifies to the appropriateness of this instructional procedure.

Glennon and Wilson (1972) have noted one caution in following this procedure which seems to be particularly apropos in corrective instruction. They suggested that it is not sufficient to merely experience concrete, pictorial and symbolic materials; they stated that

> one learns by thinking about what he is doing. And thinking can be done while a person is working with concrete things, or with pictures of things,... or with symbols. (p. 309)

In viewing mathematics learning as a generative process, Wittrock (1973) reiterated the importance of not just exposure, but active involvement, with objects, pictures and symbols.

Since general intellectual development appears to proceed from concrete to abstract, the distribution of emphasis upon actions, images and symbols should vary with the learner's developmental level, roughly indicated by age. In the context of developmental instruction, the younger the child, the greater the emphasis upon action; the older the child, the less the emphasis upon action and the greater the emphasis upon images and symbols. In the context of corrective instruction, this generalization does not appear to be appropriate. Experience in a remedial mathematics clinic has suggested the developmental level within broad content areas to be the crucial factor. Children low in a broad content area like fractions, regardless of age seem to benefit from more prolonged manipulative and image generative experiences.

Whether developmental or corrective, instruction at any age level appears to benefit from at least some concrete experiences. Archer (1972) for example, found college students to achieve and transfer more when they experienced physical materials in their mathematics instruction.

Experience in a remedial mathematics clinic has suggested the possibility (as Wittrock proposed) that the order of manipulative, pictorial and symbolic experiences is unimportant; a more apt conclusion, however, is that inclusion of special instructional experiences to promote visual images is not always necessary. Many times prolonged experiences with manipulatives appear to be sufficient for some children to succeed in later symbolic experiences. Perhaps this explains why a manipulative-symbolic instructional sequence has sometimes been found to be as effective as a manipulative-pictorial-symbolic sequence (Punn, 1974). From a pragmatic point of view, instruction might more efficiently follow a manipulative-symbolic sequence, returning to pictorial experiences if the symbolic experiences appear to be too difficult; in fact, experience in the clinic has found this procedure to be very expedient.

The implications of this paper's thesis for diagnosis in mathematics are somewhat less tenable and not as commonly applied as those for instruction. Crucial to such implications is the action-image-symbol sequence in attaining mathematics concepts. The evidence seems to support the order of this sequence; however, if there is not order (as Wittrock indicated), then manipulative, pictorial and symbolic expressions of a concept merely suggest the completeness of understanding. For example, if a student can generate equivalent fractions symbolically but not pictorially (or vice versa), then the student would be diagnosed as having only a partial understanding; corrective instruction for such a diagnosis would consist of helping the student display the appropriate manipulative, graphic or symbolic behaviors.

It is a truism that effective instruction must begin where the student is. Assuming the action-image-symbol sequence is ordered as the evidence suggests, if the student's level of attainment in a given concept is at the manipulative or visual imagery level, then it seems reasonable that instruction should start at that level. Although this is to an extent an implication for instruction, it is dependent upon a diagnosis which identifies the step in the action-image-symbol sequence where the child's concept development ceased. Therefore, the main implication for diagnosis is that a student's performance in manipulative, pictorial and symbolic tasks which embed a given concept should be examined to infer the level of attainment.

The procedure appears to be particularly useful in situations where a child has experienced previous instruction which followed an action-image-symbol sequence. Whether the instruction failed (or ceased) or the child not ready, the diagnosis suggests (a) where in the sequence instruction or concept attainment ceased, and (b) where continued or corrective instruction should begin.

Experience in a remedial mathematics clinic has suggested that examining children's performance in manipulative, pictorial and symbolic tasks is an effective procedure for determining their conceptual maturity. Although there are some exceptions at older ages and in more advance concept areas, clinic teachers seem to be able to determine cut-off points in the sequence for different conceptual areas. For example, it is not uncommon for a child to deal with division of whole numbers in manipulative and pictorial tasks, but fail utterly in symbolic tasks.

With respect to implementing the diagnostic procedure outlined above, several questions appear to be in need of attention. First, for a given concept, should diagnoses proceed from concrete to abstract or vice versa? In the former case, the diagnosis can become extremely long, while in the latter case, initial failure may have an effect upon later performance. This question is related to another unanswered question concerning the relative merits of ascending and descending diagnosis.

A second question in need of further consideration is whether it is possible to find or create manipulative (and possibly pictorial) tasks for diagnosing all mathematical concepts. Clinic teachers who have dealt with older children in the areas of decimals, percents and even fractions, for example, have particularly struggled with this question.

What effect upon performance does student unfamiliarity with the materials employed in the manipulative tasks have? This question seems to be particularly appropriate when dealing with an older child. If a child can perform symbolically but not with manipulatives, is this an indication of unfamiliarity with materials or is this symptomatic of lesser understanding?

Finally, are there ways in which manipulative tasks can be administered to more than one child at a time? Experience in a mathematics clinic has shown that students seated in a circle at desks or tables facing outward can be easily observed during manipulative tasks, but cannot themselves easily observe other children's behavior. Unfortunately, this procedure has proved feasible only for small groups.

CONCLUSION

The thesis of this paper has been that mathematical notions can be expressed through actions, visual images and symbols, which suggest levels of conceptual maturity. Support for the thesis was indicated from the theories and base research of several psychologists. The implications for diagnosis and instruction were then explored; and finally, several questions relative to the implied diagnostic procedure were presented.

The ideas presented in this paper will hopefully provide a framework for helping those children who experience learning difficulties in mathematics. At the outset, it was the writer's intent to challenge the reader with a thesis whose implications related diagnosis and corrective instruction. The real challenge of this paper, however, is to stimulate discussion and research which may ultimately lead to improvement in mathematics education.

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Chapter 4

RESEARCH AND DEVELOPMENT RELATED TO LEARNING ABOUT NUMERALS FOR WHOLE NUMBERS: IMPLICATIONS CLASSROOM/RESOURCE ROOM/CLINIC

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INTRODUCTION

I find when working with children in our clinic and in classrooms that it is frequently the case that a child can do specific numeral-related tasks but does not use the numerals as if he or she understood what the symbols represent. This is noteworthy not only when the child computes but also when he or she uses numbers in other contexts. In my work with teachers, I find the term "place value" is used to designate any task which has something to do with what numerals mean, even though place value concepts are only a subset of the set of numeration related concepts. I also find instructional activities are similarly limited ... relatively few exemplars are used.

There are research studies that suggest we need to take a closer look at how children learn and use numerals for whole numbers. Uprichard and Collura (1977) report a clinical study in which an emphasis on mathematical structures such as place value appeared to facilitate not only concepts, but computational skills as well. Charles Smith (1969) noted, when reporting his study involving subtraction computation, that "Pupils proficient on the renaming subtests had less difficulty in subtracting than those less proficient." He found that low achievers on the Iowa Tests of Basic Skills were generally unable to apply decimal numeration principles, and concluded that a greater emphasis needs to be placed on building pupil ability to apply principles of decimal numeration within subtraction computation. However, it should also be pointed out that other researchers, such as DeFlandre (1975) and Fishell (1975), both of whom focused on instructional in numeration and its relation to skill in computation, did not find such understandings necessarily leading to measureable change in computational skill.

Another reason to consider numeration as a topic worthy of attention is the wide-spread use of hand-held calculators, and the increased need for estimating skill which has resulted. A major prerequisite to estimating skill is understanding of numerals for whole numbers.

GENERAL COMMENTS ON RESEARCH

Children's learning about numerals for whole numbers and their use of numeration principles when doing specific tasks, such as computing or estimating, has not been researched carefully or thoroughly. Much of the

research that has been done has focused on learning non-decimal numeration systems, and on the effects of such instruction. We need to take a much broader look and to consider research across a wider range of issues. In order to do this, we need to structure the field for investigation and to demonstrate the relatedness of the many aspects of teaching and learning numeration. In short, we need theoretical structures which will point more clearly to specific investigations and permit us to relate one study with another more easily.

It is not surprising that research in the area has tended to be scattered. Many studies have lacked a clear focus, and even well-done studies have been difficult to relate to one another in terms of implications for instruction.

One difficulty is a lack of concensus as to the definitions of terms, specific areas to be investigated, issues involved, and how these relate to one another. There is much confusion with regard to terminology, especially in the professional literature of other fields of inquiry which bear directly upon our topic. For example, you will find expressions like "recite numerals," or references to oral names which do not make clear what sort of oral name is being used. If we are more careful with our choice of terms, we are apt to find this a much more complex area for research and for diagnosis and instruction than many of us have assumed. Perhaps we should not be surprised that our friends from other fields, such as psychology, need help in analyzing the area and in using terminology so it is understood and can be applied by math educators.

I am using the term "number" to refer to the idea or concept of number, the term "whole number" to refer one of a specific set of numbers, and the term "number system" to a system including a set of numbers, operations defined on the set of numbers, and specific properties of those operations. However, I am using the term "numeral" to refer to a standard written symbol made up of figures or digits, and the term "numeration system" to a system for producing numerals. Such systems include a set of figures or digits, and rules for applying concepts such as grouping, place value, addition, and multiplication. There are advantages that accrue when we distinguish between mathematical ideas like number and the symbols we use to write about them. Most notably, they are separate things to be understood. Admittedly, many of the errors children make in mathematics result from a lack of understanding of the math itself, as noted by Hamrick and McKillip (1976). But they also concluded from their study that other errors resulted from a lack of understanding of the symbols used.

Perhaps one of the reasons numeration has not been looked at carefully by math educators is that we stand in a very different position than the young child of 6 or 7 who is learning about numerals for whole numbers. As adult teachers of mathematics, we process mathematical information with comparative ease. Yet Krutetskii (See Kilpatrick, 1977) points out that one of the abilities needed to do this is the ability to think in mathematical symbols: both in numerical symbols and letter symbols. We are accustomed to using numerals in thinking about mathematics. But the young child does

not yet have the ability to think in mathematical symbols, to use them while focusing on something else. The symbol itself is, at least to some extent, the object of his study. Ginsburg (1977) emphasizes that children search for meaning, for underlying structures even within our systems for naming numbers, a search that sometimes leads to incorrect assumptions for the systems are not thoroughly consistent. My colleague John Elliott, a developmental psychologist concerned with cognition, refers to a symbol system such as a numeration system as a logical system, for it is a system of organization and order with specific conventions; he is concerned that we may be expecting children to learn such logical systems when they have not developed to the point they possess the required logical abilities. Surely, then, we have ample warrant for being concerned with children's understanding of numerals as well as their understanding of numbers and operations on numbers.

But numerals are not the only way we represent whole numbers. Systems of oral number names and the system of written number words parallel the system of numerals. Though we may introduce these systems of representation to children at very nearly the same time, they are different systems. For example, we record the numeral 56 and we mean 5 tens 6 ones, but when we say the standard number word, we do not say "5 tens 6 ones;" rather, we say "fifty-six." We do not even say "5 tens 6;" we say "fifty-six." Philologists tell us that eleven derives from words meaning "one left over," twelve from words meaning "two over ten," and hundred from a term meaning "ten times" (Eves, 1964, p. 8), but such root meanings are, of course, not obvious to children. The systems of representation are obviously related, but they are different. With numbers like "1 ten 3," which we call "thirteen," they are very different indeed. Yet, during assessment and instruction, we often use these systems interchangeably, assuming the child can translate easily from one system to the other.

One way of looking at number concepts and the systems of representation is suggested in figure 1.

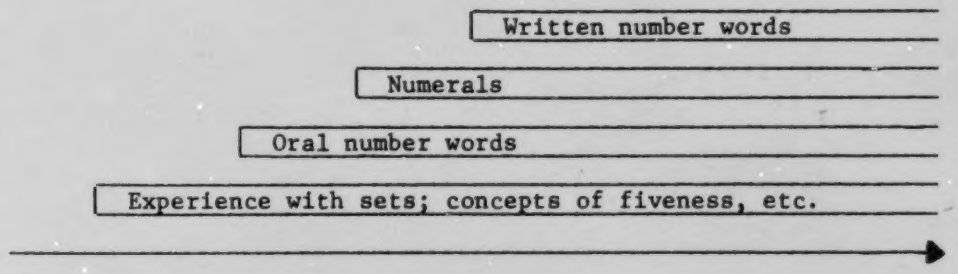

Figure 1. Systems for Representing Whole Numbers

Note that experience with sets occurs first, number concepts are developed from those experiences. Such experiences continue and interrelate with later experiences with representation systems. Children normally do encounter oral number names to twenty or even to a hundred before the corresponding numerals are introduced, for the young child is taught to count by rote. Ginsburg

(1977) reminds us that when a child begins to write numerals, he sometimes makes the error of writing numerals in the same way he says corresponding number words. Twenty three might be written 203 as a straight forward translation from words, in accordance with a system of rules which is partly of his own invention.

Figure 1 is an oversimplification, especially for numbers greater than 9, for multi-digit numerals involve concepts deriving from more complex experiences with sets, such as forming sets of sets. They also involve concepts which do not derive from underlying experiences with sets, such as the concept of value assigned to place.

However, it is possible for a child to make rather straightforward associations of sets with oral words and with numerals for not only 0-9, but for larger numbers as well. Children who make such straightforward associations can enumerate a set, can tell you how many it contains, and can sometimes write the multi-digit numeral for that number, but they do not understand the structure of the numeral in a way which will permit them to use it creatively when renaming and when estimating. Such children learn about multi-digit numerals much as children learn to read so-called sight words. They work with multi-digit numerals as if they were single digits. I have found this happening most often with the teens, with powers of ten, and with multiples of powers of ten. This may be happening when a child writes 10 as the numeral for a set of ten before instruction in grouping and place value. It may also be happening when a child writes 605 for 65, thinking of 60 as a symbol for 6 tens. When children add but fail to rename, they often write answers which crowd two digits into one place, sometimes interpreting their answer correctly without recognizing it makes no sense as a numeral. For example in:

$$\begin{array}{r} 647 \\ +289 \\ \hline 81216 \end{array}$$

the 81216 is interpreted as 8 hundreds, 12 tens, and 16 ones.

Each way of representing numbers is a system with its own structure. For example, within the sequence of oral number words there are patterns which children frequently observe.

> Alan was counting by rote. I stopped him after 49 and asked, "What comes next?" "Fifty," he said. "How do you know?" Immediately he replied, "Cause you go 1, 2, 3, 4, 5.

Most children probably observe these patterns on their own, but I have talked with children who were not aware of them. A number of questions occur to me:

What do the children we work with understand about each of these systems of representation as a system?

We use words as we talk about numerals. When does this confuse the child?

Do we help children find patterns within each of the systems, or do we expect children to rely only on arbitrary associations or the rules they happen to construct?

Which of a system's concepts do children need to understand? And when?

When are they able to learn them?

If we "read the numerals" differently than "standard" number words (e.g., 26 as "2 tens 6 ones" instead of "twenty-six"), does this confuse the child when it comes to understanding the systems of word names?

Another difficulty when thinking about teaching and learning numeration is deciding what is meant by "understanding," and by "meaning" in this context. Barr (1978) speaks of instruction in the meaning of 2-digit symbols. Fishell (1975) speaks of understanding a decimal numeration system. There are tests called tests of understandings. And so it goes.

What do we mean when we talk of a child knowing or understanding something like place value? From existing research we could abstract a number of operational definitions, and these would vary from specified scores on paper-and-pencil tests to tasks with manipulatives and verbal explanations. Each such behavior lets us infer, with a greater or lesser degree of confidence, that the child understands; he knows. But I have seen children who learned to do many of these specific tasks but were not able to relate the idea to other ideas or to use the information at all; I have seen this happen too often to have much confidence in many of the tasks we give children for evidence of understanding place value or numeration. It is too much to hope the cognitivists and behaviorists will get together on what it means to understand something; but we hope that, in time, they will together give us new insights. Meanwhile, we need to proceed with caution and at least build into research studies a variety of tasks. John Wilson's (1976) grid of possible within-across exemplar class translations from stimulus to response is helpful for planning a variety of tasks, but even these do not explore the full range of sensory modalities that may need to be looked at if we are to find out what a child understands.

One thing we can do to help structure the field of study is to agree on categories of numerals for investigation. I suggest three rather self-evident categories; but in research and in teaching, even these distinctions are not always carefully drawn:

1. One-digit numerals (0-9)
2. Two-digit numerals (10-99)
3. Three-or-more-digit numerals (100 +)

One-digit numerals are associated with number concepts of "order," "cardinality," and "many-can-be-shown-with-one-digit." The association of number and number name is completely arbitrary; no system of numeration is obvious.

Two-digit numerals do follow a systematic pattern, one which involves new concepts such as "number of groups," "value assigned to a place," "addition," and "multiplication." However, place-value concepts are not generalized. Children often think of tens as left and ones as right. It has been encouraging to see a few research studies focusing on 2-digit numerals; those by Rathmell (1973) and Barr (1978) are notable examples. Scrivens (1978) concluded that the third graders he studied had an incomplete understanding of place value beyond 2-digit numerals, and needed more work with tens and ones before extensive work with larger numbers. Smith (1973) concluded that children are too often introduced to the concept of hundreds before adequately understanding the tens in two-digit numerals. These studies recognize 2-digit numerals as a specific category of numerals for research and for instruction; more such studies are to be encouraged.

In contrast to learning about two-digit numerals, when children learn about numerals with three-or-more digits, place-value ideas need to be generalized so the system can be extended indefinitely. John Wilson (1976) has recognized the difference between 2-digit numerals and 3-or-more-digit numerals in his structures of content objectives, specifying additional ideas a child needs to understand when working with numerals of 100 or greater.

For example, I remember Alan, who told me that in the numeral 106, the one is in the tens place and the 6 is in the ones place (he thought of tens as left and ones as right). I asked him, "Where is the zero?" and he said "The zero is in the zero's place." Asked to point to the ones in 2-digit numerals like 26 and 74, Dana said she could not, for "There aren't any ones." Pointing to the digit on the left, she said it was hundreds; and then pointing to the other digit she said it was tens. "There aren't any ones." Dana had recently been introduced to three-digit numerals, viewed the digit on the left as hundreds and the next one as tens. She had not yet come to understand the general pattern for assigning place values.

DIAGNOSIS

"How do we go about diagnosing what a child understands about numeration?" To begin, we need to consider the full range of abilities and concepts involved in understanding numerals for whole numbers. I want to discuss these and some assessment tasks I have found to be helpful.

A number of attempts have been made to list the concepts and abilities involved in learning about numerals, though such lists sometimes include only content ideas. Examples of lists of concepts and abilities can be found in recently published books such as the following: Math H.E.L.P. and One Step at a Time, in Ronshausen's (1978) recent article in The Arithmetic Teacher entitled "Introducing Place Value," and in certain of the content structures constructed by John Wilson (1976). I will not now attempt to present the definitive listing of concepts and abilities that might be considered during diagnosis, but I do want to comment about several of them.

Of course, rote counting is involved as is rational counting, and these are skills which entering kindergarteners typically possess. Much research has focused on a child's understanding of the number of a set; these have been mostly Piagetian studies. But at present, the relation of conservation of number to the introduction of numerals is not at all clear. Generally, it has been believed that conservation of number is necessary for the development of numerical understanding, but Mpiangu and Gentile (1975) concluded from their study that "conservation is not necessary for mathematical reasoning." They felt it "would be more profitable to consider arithmetic and conservation of number as conceptualizations that develop simultaneously." I realize this is a very controversial area, and I mention it here only to point out that the place of conservation of number, per se, as a concept or ability needed for children to learn about numerals is, as yet, unclear.

Though the earliest written symbols for number were probably tally marks (Wilder, 1968), ours is a cipherized system of numeration; therefore, it is necessary for children to understand that many can be shown with one symbol, a digit. The face value of each of the ten digits tells how many things. Both research and general practice in our culture support introduction or oral number names before introducing the digits. For the numbers 0-9, the translation is straightforward; difficulties usually begin with the set of ten and above, with the introduction of 2-digit numerals.

For these numerals, it is necessary for children to understand not only the concept of a set, but also the concept of a set of sets. Within our cipherized numeration system, this means children need to know not only that many of ones can be shown with one digit, but also that many of tens can be shown with one digit, many of hundreds can be shown with one digit, and so forth.

Use of a base collection is difficult for some children; they can count single objects and can group objects, but they cannot count groups of objects. Mary could group objects into sets of five and she could count the number in each set, but she was confused when asked how many sets she had made. Not surprisingly, when shown five tens and two ones with base ten blocks, and asked how many tens, she said "Seven," though she had previously identified each piece of wood correctly. (Children who do not understand the grouping idea, who can count sets of ten, sometimes do not know what to do with the singles left over.) Both Scrivens (1968) and Underwood (1977) concluded from their research activities that the grouping idea is more of a problem for children learning numeration than is the place value idea.

Of course, our numeration system is a place value system, and that is why it lends itself to the processes of arithmetic. Neugebauer (1957, p. 5) considers the invention of place value notation as "undoubtedly one of the most fertile inventions of humanity," an invention that can "be properly compared with the invention of the alphabet as contrasted to the use of picture-signs intended to convey a direct representation of the concept in question."

What do children need to know about place value? They need to know that values, powers of ten, are assigned to each position. Also, they

should understand that the position of a digit determines its value; or, as Karen Fuson has said, "What the value of a digit is, depends on where it lives." (1977) A related idea, sometimes overlooked, is the fact that only one digit can be assigned to a position or place. When children do not keep this in mind, a variety of errors occur.

Zero is another critical concept. Historically, the digit "zero" was used in several numeration systems as a device to show a place not used, but sometimes this practice was incomplete. For example, the Babylonian zero was apparently used only medially and not at the right of a numeral; 605 and 65 could be distinguished, but not 650 and 65 (from Wilder, 1968). For such distinctions one had to rely on the context, and this has resulted in some interesting dilemmas for archeologists. For us, zero is not only a place holding device, it also names a number. In fact, all ten digits serve both functions; they are place holding devices and they name a number. However, many children do not understand that this is true. For them, a zero is merely "nothing." There is a difference between understanding the zero in 204 as "There is nothing in the tens," and understanding it as "There are no tens."

Many other concepts are involved in learning about numerals for whole numbers. For example, some understanding of both addition and multiplication are required even for two-digit numerals. As John Wilson notes, the value of a digit is the product of its face and place values, and the value of a multi-digit numeral is the sum of the product values for each digit (Wilson, 1976). There are other abilities and concepts involved in learning about numerals, ones not considered as often by math educators when diagnosing children.

One area of such abilities is spatial orientation and imaging. We have left such concerns to psychologists and neurologists, but when we plan instruction involving written symbols, we would do well to consider carefully what they are learning. This is especially true for instruction in multi-digit numerals. We are reminded of the difference between one-digit numerals and numerals with two-or-more digits by studies such as those by Luria (1969), who found that many individuals who become brain damaged retain the ability to perform simple serial computations within the limits of the first ten numbers but are not able to perform more complex operations. He concluded that the basic characteristic of acalculia is impairment of the number concept related to breakdown of thought, which leads to deterioration of number beyond simple numerical relatedness and includes the accompanying loss of place value. We also know from studies such as the one by Guay and McDaniel (1977), that high achievers in mathematics have greater spatial ability than low achievers.

Left-right discrimination is a part of this area, but only part. Children who can identify objects to the left and to the right in relation to a specific object in a sequence, sometimes have great difficulty learning the pattern of assigned place values and the relation of one value to another, either to the left or to the right.

Reversals are often a problem as children begin to write numerals, both the mirror image of a digit and the exchange of two adjacent digits. This phenomenon has been examined more extensively by researchers concerned with language education than by those concerned with mathematics education. Moyer and Newcomer (1977) note that with reference to making the mirror image of a single letter (and we might assume some relevance for making the mirror image of a single digit), two schools of thought exist. One school suggests the difficulty is caused by immaturity of development, the result of neurological malfunctioning, confused lateral dominance; but apparently this has not been verified empirically. The other school suggests there is evidence maturity is not a factor; directionality is a learned cognitive skill. Moyer and Newcomer (1977) suggest that young children use symbols to identify and name things, and it is possible that for them spatial orientation is not a significant feature for identifying and naming things. The child has to learn the importance of this feature.

Interestingly, I recently saw a second grader reversing his 6 so it looked very much like his 2. I dared to show him how a 6 is made and that it is very different than his 2, and he volunteered the fact that he used to make them my way but his friend showed him how to make them the way he was making them. (The next time I observed him making a 6, he made it correctly.)

Stanley (1976) studied children's processing of digits, but they were children with specific reading disabilities. He found that the rounded digits, 3, 6, 8, and 9 were most frequently reversed.

Piaget and Inhelder have hypothesized that two-dimensional reference systems do not develop until the period of concrete operational thought, and the findings of Pufal and Shaw (1973) support their hypothesis. It has also been observed that many 7-year-olds can read maps (i.e., use the symbol system to translate symbols into their referents), but they cannot make maps, they cannot locate and record the symbols in the appropriate place. I am not aware that the implication of this for children being introduced to multi-digit numerals has ever been examined, but it could make a very interesting study.

Another ability which seems to be important in learning about numerals is verbal ability. We know that words are not always associated with correct concepts. Sometimes, the words "more," "greater," and "less" do not mean exactly the same thing to a child as they do to us, but we use these terms regularly during math instruction, even when talking about place values. Order and equivalence relations are certainly involved in learning about numerals, and we tend to assume children have these concepts well in hand long before being introduced to multi-digit numerals. But is this true? Walker (1975) found that initially, a child does not use such relational terms in a true comparative sense. Rather, they are used in a nominal sense. The child who uses the term "more" in a nominal sense has difficulty understanding it is possible for the digit in the tens place to be worth <u>more</u> than the digit in the ones place, and at the same time be worth <u>less</u> than the digit in the hundreds place. Also consider the situation when more milk

is added to Joe's glass, but less is added to Mary's glass than to Joe's. McNeill notes in The Acquisition of Language (1970) that some children think "less" means "more."

Word order creates some interesting problems, too. I found that what June, a second grader, understood as the meaning of "2 more than," is what I would call "more than 2." (Sets which were "2 more than" a set of 3 included sets of 3, 4, 5, etc.) We have already noted that the word-symbol systems for naming numbers are different from the numeral system, thereby complicating instruction and learning substantially. I think we need to ask: To what extent are children taught what multi-digit numerals mean (and to what extent should children be taught what multi-digit numerals mean) before they are capable of concrete-operational thought?

If our diagnostic activities with a child are going to get at abilities and concepts such as those which have been described, we will have to have assessment tasks which are different than the usual tasks we find on available instruments. Let me share with you just a few tasks I have found useful in diagnosis.

1. Show: 27 objects (alike)

   Say: "If you want to tell your friend how many are here, is there anything you could do so you would not have to count them one-by-one? ...not have to count them 1, 2, 3, ...?

Some children do not group, others form groups but not of the same number, and others form groups of the same number. For those who do not group, I ask "Would this help?" and start grouping by 2's. If more guidance is needed, I say "Have you ever counted, 2, 3, 6 ....?" If a child does form groups of the same number, I ask "Could you do it a different way?" Note what a child does with left-over singles. Some children count the singles as if they were groups.

2. Show: 27 objects (alike)

   Say: "Make a set of 5 _____ "
   (objects)

   Say: "Now make another set which is 3 greater than this set." (Point to set child made.)

If a child cannot do this, say "Make another set which is one greater than this set. If he does, ask for "3 greater" again.

Remove the sets, except the set of 5.

Say: "Now make a set which is 3 times greater than this set."

If the child cannot do this, say, "Show a set which is 2 times greater than this set."

Children who can easily make a set of 5 often have difficulty making a set which is 3 greater than the set of 5; they make a set of 3. The same results follow with "3 more than." The number name is related to the size of the set, and not to the size of the difference of two sets.

For "3 greater than," other children make a set of 4, because "4 is greater than 3." For them, "3 greater than" and "greater than 3" mean the same.

Our instruction often focuses on "greater than;" should we also be focusing on "how many greater than?"

In order to generalize the numeration pattern, children need to be able to think about "N times greater." In the numeral 22, one digit is "ten times greater" in value than the other.

3. Show: 5 bundles of ten sticks, and 15 single sticks.

   Say: "What would be a good way to show me 26?"

   After the child responds ....

   Say: "How do you know that is 26?"
   "Why is the 2 here (point to the digit 2) and not here (point to the 6)?

Dana readily picked out two bundles of ten and six sticks. In response to the question, "Why is the 2 here?" she said, "Because the ones come first. My teacher says 'Ones come first.'" Asked, "Which is the ones?" Dana pointed to the single sticks. "How many ones?" "Six." I pointed to the numeral again and asked "Which is ones?" "Two" she said. For Dana, numerals were not a record of observation with sticks or blocks.

4. Show: 17 and 71

   Say: "How are they alike?"
   "How are they different?"
   "Which shows more?" .... "Why?"

I find this item provides a very effective context for discussing the attributes of a numeral with a child and it evokes some interesting comments. Some children admit, quite frankly, they do not know which shows more.

5. Show: Pictures of base ten blocks and a 3-digit numeral, as follows:
   a. No zero in numeral, blocks in sequence
   b. With zero in numeral, blocks in sequence (See Appendix A)
   c. No zero in numeral, blocks out of sequence. (See Appendix B)
   d. With zero in numeral, blocks out of sequence.

   For each say: "Draw rings around enough blocks to show as many as this." (Point to the numeral.)

After each response say: "Now read this number for me."

I find, not surprisingly, that children who are not consistently able to name the value assigned each place have difficulty relating a digit to the specific piece of wood as they proceed through the tasks. For such children, the zero may not be a problem at all. This item, and a 4-digit variation, do appear to help some children learn to read the numerals.

6. Show: Problems as follows:

   a. 23 + 41 = ____

   b. 35 + 27 = ____

   For each say: "See if you can figure out the answer in your head, without writing out the problem."

   After each response, say: "How did you add it?"

My interest with items such as this is not in addition skill, as such, but rather what a child understands and how he can use what he knows about numerals. This is an especially interesting question to pose to a child who has not been introduced to addition with regrouping. I have been pleased to see the number of children who do add ones to ones and tens to tens when presented with the horizontal format, even when they are not able to read all of the numerals correctly. But an oral response to 36 + 27 as 513 is not unusual.

There are a number of sources one can turn to, to develop items for assessing numeration. Brownell's (1941) article in the 16th NCTM Yearbook included items like "Rearrange the figures in the number 51,937 to make the smallest possible number," and "In the number 7,463, what is the largest possible number of hundreds?" More current are selected items in the Maryland Diagnostic Arithmetic Test and Interview Protocols (Wilson and Sadowski, 1976) and Wiebe's (1977) Place Value Knowledge Survey, which includes a number of tasks using quite a variety of materials.

INSTRUCTION

First of all, we ought to recognize that learning about our numeration systems takes time ... years (Ginsburg 1977). We should not expect children to apply mature concepts of numeration quickly; and when we do introduce computational procedures, we need to carefully guide children in their application of numeration principles. Apparently, numeration is more complex and more difficult for many children than many have assumed.

Also, we should be careful to teach addition with sums no greater than 9 before introducing 2-digit numerals. John Wilson makes a point of this in the hierarchical ordering of his structures of content objectives. If children do not understand the operation of addition, they cannot understand 24 as 20 + 4 except as an arbitrary manipulation of symbols you do when the teacher gives you blanks to fill in.

Multiplication and division with one-digit numbers should also precede work with ones and tens. Underwood (1977) has made this point, and I believe he is correct. Symbolism need not be introduced, but children need to understand multiplication as we find it suggested in statements like "2 three" and in "5 tens;" and they need to understand division as we find it suggested in "How many 2's in 6?" and in "How many tens in 34?" The latter even includes the notion of a remainder.

This leads very naturally to an emphasis on the grouping idea, including (eventually) groups of groups. Teachers have not led children to examine such structures until rather recently, and even now it is not clear that groups of groups receive much attention. Certain stick-bundling activities, work with sets of blocks representing powers of ten (and other bases too), and chip-trading activities involve such structures. What is not yet clear is how such activities should be sequentially related to the introduction of oral number names and written multi-digit numerals.

There is considerable controversy at present concerning the introduction of 2-digit numerals from 10 or 20 to 99. The controversy concerns the sequencing of two types of instructional activity: 1) helping the child understand the two digits in terms of the number of groups of ten and the number of units, and 2) helping the child state the standard oral number names in sequence. As far as the numbers 20-99 are concerned, there seems to be a bit of a trend toward teaching the number names for 20-99 through counting activities before teaching the meaning of these particular numerals. If we consider instruction in language as did Hickerson (1952, 1959), we will probably want children to have experiences with sets of 20-100 which they count with the standard oral names, to be followed by introducing the symbols to be read with their standard names, to be followed by writing the symbols. Meaning for the symbols could be developed during the original counting activity (as sets of ten are completed successively and later as numerals are analyzed in terms of tens and ones. An emphasis on counting by tens is likely to help with the often difficult translations back and forth between the standard oral name and the grouping number name for multiples of ten (e.g., "Twenty" and "Two Tens"). Experiences with both rote and rational counting by tens can be provided, and both standard and grouping number names can be used in these experiences.

The familiar counting frame is an excellent exemplar for 2-digit numeration, and it can be used in varied ways. One variation is to count not only with standard oral names but also with grouping number names (e.g., nine, ten, ten and one, ten and two ... (later) ... 5 tens, 5 tens and one, 5 tens and 2, etc.). The other variation is to introduce activity which bridges carefully to the numeral. As one child counts slowly with standard number names, other children can show the numeral with digit cards, or perhaps they can write it. Similarly, as a child counts with grouping number names, other children can show the corresponding numeral.

As 3-digit numerals are introduced, another important question faces us. Is it easier for a child to understand the general principle of numeration from concrete experiences of grouping with a variety of bases, and then apply this understanding to base ten; or to learn the specific pattern of

base ten, and then develop a more generalized concept of power? Dienes (1966) argues for the former, but many have concluded from research which has been done on teaching non-decimal number bases that direct teaching of base ten is as effective or better than teaching non-decimal bases, per se. Many of the writings of Dienes, and instructional activities such as chip trading games, focus primarily on helping a child abstract the structure of the system. The effects of chip trading games on learning decimal numeration is a proper object of study, but such studies should not be confused with studies which are typical of those focusing on instruction in non-decimal bases.

Instruction in other number bases has been the focus on a number of studies which, taken as a whole, do not give much support for teaching non-decimal bases in order to help children perform more successfully with decimal numeration. Such studies tend toward non-significant findings. When considering such studies it might be helpful to distinguish between numeration systems with the Hindu-Arabic pattern and other systems on the one hand, and decimal and non-decimal bases on the other hand. Four categories are identified in figure 2.

STRUCTURE	BASE: Decimal	BASE: Non-Decimal
Hindu-Arabic	A. Our Numeration	B. Our numeration adapted to other bases
Other Systems	C. Egyptian ⋮	D. Mayan ⋮

Figure 2: Categories of Numeration Systems

Such a categorization may help us organize the results of such research in a way that points more clearly to implications for instruction, e.g., in Scrivens' (1968) study, A only was more effective than B followed by A, but he also found instruction in C more effective than B. McCormick (1965) found a trend toward A only as being more effective than A and B combined.

Another problem area for instruction is the introduction of regrouping. By introducing addition and subtraction of multi-digit numbers without regrouping, and practicing that skill for a year or so (as is typical in our primary programs) children rather typically come to view numerals within examples as a collection of digits to be examined in vertical columns.

What they know or may have known about multi-digit numerals tends to be set aside; the idea that in subtraction the minuend is larger than the subtrahend is irrelevant, for they only look at pairs of digits. Perhaps it would make more sense to use what children are learning about numerals as they move into multi-digit addition and subtraction including, from the outset, both examples with no regrouping and examples with regrouping. Of course, this should be done only by using sticks, base blocks, etc. quite extensively, and delaying introduction of the written algorithm. Answers to given examples would be worked out with materials initially, and the more efficient paper-and-pencil procedures would come later. Though such a suggestion is at variance with current practices in the primary classrooms of American schools, it is not really such a unique idea; and it does potentially allow a child to reinforce what he has learned about numerals and see how that knowledge can be applied, rather than to forget that knowledge and/or come to view it as irrelevant with regard to computational procedures.

As we provide children with 3-dimensional exemplars for numeration concepts, there are several cautions we do well to keep in mind. For one thing, we need to be alert to specific limitations of individual children. A child who does well with gross motor activities but less well with fine motor activities should not be asked to bundle sticks; he may be able to do a similar activity with base blocks or an over-sized version of them. Knaupp's (1970) study reminds us that whenever materials like sticks and blocks are used for learning about numeration, children (except high IQ children) appear to profit from teacher demonstrations along with their own manipulation of the materials. Of interest also is Wiebe's (1977) finding that during such concrete activities, the amount of pupil verbalization did not have a significant effect on learning the concept of place value.

Steinberg and Anderson (1973) remind us of the presence of distractors in the physical models we use to exemplify numeration concepts. It is true that concrete materials are not helpful just because they are concrete; all such materials are imperfect aids to instruction in that they have attributes irrelevant to the idea being exemplified ... color or shape or woodenness or some such. They also vary in the extent to which they exemplify the set of concepts a child needs in order to understand and use numerals.

A categorization of instructional aids may suggest an appropriate order for introducing numeration exemplars, and one such categorization is as follows: (Ashlock, 1976)

Category 1 "Objects ... placed in sets of ten:" examples include stacking chips, bundling sticks, etc. As valuable as such activities may be, we should not be surprised when children do not abstract place value ideas; place value concepts are not exemplified in the activities.

Category 2 "Objects ... traded for a larger object which looks like a set of single objects glued together:" examples include "sticks used in a way where there is no bundling but only trading for already-made bundles," base ten blocks which are scored, etc. Knaupp (1970) inferred

from his study that the block model for numeration was more interesting and enjoyable for children than the stick model, but we should keep in mind that even activities with block models may not exemplify place value concepts. The value of the block is not determined by where it is placed.

Category 3 "Objects which are traded for a single object the same size and shape as a unit object, but are distinguished from the unit object by color or by both color and place:" examples include many chip-trading activities, a computing abacus with different colors, etc. The presence of color makes it impossible to tell if a child has understood and is or is not using the idea that place determines value. And for this reason, some have said that such activities should not be used. I think they have their place within a sequence of activities but we should not abandon the child to symbols after Category 3 activities.

Category 4 "Objects traded for a single object identical to the unit object but distinguished from the unit object only by the placement of the object:" examples include many place-value charts and devices, a computing abacus with discs of only one color, chip trading games with one color, etc. The child who uses these aids correctly is applying place value ideas and may be ready for more extensive work with numerals alone. "Instruction in numeration often involves children with aids in Category 4 too early, and children merely learn complex mechanical procedures for getting answers. The use of varied instructional aids in a sequence similar to that suggested ... may do much in helping children relate numerals to appropriate collections in their environment."

CONCLUDING COMMENTS

I raise again the question: What are we expecting of children in the primary grades with reference to numeration? Arbitrary but unrelated associations? Specific fill-in-the blank tasks? What should we be expecting?

Vygotsky observed that "the young child thinks by remembering, an adolescent remembers by thinking," (Luria, 1976) a statement reflecting thinking which is similar to what we find in Skemp's (1976) distinction between instrumental and relational understanding. The young child remembers a specific fact or rule then uses it, whereas the older child uses his understanding of relationships to bring to mind specific facts and rules.

In order for a child to use numerals for whole numbers effectively in computing, in estimating, and in problem-solving generally, the child must have a relational understanding of such numerals. If we are wise, we will carefully plan our diagnoses, our prescriptions, and our instruction to help children go beyond an instrumental understanding of numerals to a relational understanding.

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Chapter 5

A THEORETICAL STUDY OF THE ORGANIZATION OF BASIC ADDITION FACTS FOR MEMORIZATION

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"If ones knowledge about diagnosis and instruction in mathematics is to improve, then we collectively must be more imaginative in our approach to inquiry" (Romberg and Uprichard, 1977, p. 1). In this paper, I suggest one technique of research which could be used and discuss the first step in this process.

After reviewing the many papers presented at the first three national conferences on remedial mathematics, I am convinced that sequencing of mathematics into hierarchies is of central importance. From this conceptual basis, a four step procedure is being suggested. First, logical analysis is conducted on the mathematical content. Second, this analysis needs to be verified by relating the analysis with the actual learning patterns of children. The third step will be to create diagnostic and corrective teaching procedures that are compatible with the logical analysis. Last, an experiment will need to be carried out to evaluate the effectiveness of the created procedures.

Utilizing this four step procedure, I discuss the initial logical analysis of a very troublesome area, the addition basic facts. It is hoped that this analysis will lead to insight into such questions as: Why do children have difficulty with the addition basic facts?; Why do children have difficulty memorizing the basic addition facts?; and, Why do children arrive in fifth, sixth or even higher grades and still do not have the basic addition facts memorized?

Examining the words "basic," "addition" and "fact" indicates interest in the fundamental addition combinations that are the base upon which all other addition examples are constructed. Since there are ten basic digits and beyond the basic ten digits, all numbers are written using two digits to express one number concept, then all basic facts would include one digit's addends with sum of one or two digit numbers only. Thus, addition examples with addends which have more than one digit are not basic facts. For example,

$$\begin{array}{r} 12 \\ +\ 3 \\ \hline \end{array}$$

depends upon two basic addition facts, and the concept of place value. The two basic facts in this specific example are two add three and one add zero and this is not a basic addition fact. Therefore, a basic addition fact is a one place whole number added with a one place whole number in order to obtain a one or two place whole number which is called the sum. All of the basic addition facts can be located by using an addition table. Place all possible one place addends on both the horizontal and vertical

axis of a rectangular table. Writing sums in each of the cells of the table created by the intersection of the addends on the horizontal and vertical axis generates what we call the addition table. The basic addition facts as defined above and developed through the table approach becomes as indicated in Table 1.

Table 1

Table of Basic Addition Facts

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Given the addition table, there are two basic techniques which can be used to organize the basic addition facts. One method, which seems to be most popular in elementary school textbooks, is to organize the basic facts according to sums. What are all of the basic addition facts which have a sum of ten? Some textbooks call all of the combinations for a given sum a "family of numbers." For example, the basic addition facts that have a sum of ten (the "family of ten") are:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{+9} & \underline{+8} & \underline{+7} & \underline{+6} & \underline{+5} & \underline{+4} & \underline{+3} & \underline{+2} & \underline{+1} \end{array}$$

There is one example with a sum of zero, $\begin{array}{c} 0 \\ \underline{+0} \end{array}$; two examples with a sum of one $\begin{array}{c} 1 \\ \underline{+1} \end{array}$ and $\begin{array}{c} 0 \\ \underline{+1} \end{array}$; three examples with the sum of two $\begin{array}{c} 1 \\ \underline{+1} \end{array}$, $\begin{array}{c} 2 \\ \underline{+0} \end{array}$, $\begin{array}{c} 0 \\ \underline{+2} \end{array}$; etc. Table 2 shows the numbers of examples in each "family."

Table 2

Numbers of Examples in Each
Basic Family of Numbers

Sums	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Number of Examples	1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1

From Table 2, it becomes apparent that all basic addition facts have been named. Following the pattern of the table, every cell to the left or right of the extended table would contain a zero for the number of examples. There cannot be any additional basic addition facts.

It becomes rather easy for young children to verify the basic facts as organized above by using sets of blocks, straws, or plastic spoons as counters. For example, if seven blocks are randomly put on a table in front of a child, he can easily separate the seven blocks into two separate sets and locate all of the possible combinations of two sets for the number seven (See Figure 1).

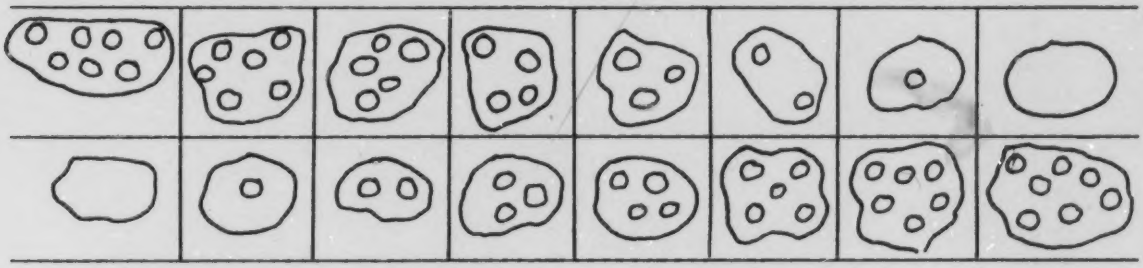


Figure 1

Combinations of Seven Objects

This approach seems logical to help young children model the two sets of addends for a given sum. However, it does not seem to help them with the concept of "addition" and it does not help them memorize the basic addition facts. It does help children develop a model to justify why a certain combination of two addends provides a unique sum. Most children who are taught by this approach depend upon counting for finding sums and become confused about the difference between addition and counting. This method of organizing the basic addition facts is beneficial for development of addition through models but is of little value in helping memorize the basic addition facts. The common element in each of the examples of a particular family of numbers is the common sum. In each addition example the sum is missing so that the respondent does not visually view the common element. It becomes difficult for the student to use this technique with an unseen clue.

Now let us examine the basic addition facts from a structural analysis of the basic properties of mathematics. Carefully examine the following nineteen examples.

0	0	0	0	0	0	0	0	0	0	9	8	7	6	5	4	3	2	1
+0	+1	+2	+3	+4	+5	+6	+7	+8	+9	+0	+0	+0	+0	+0	+0	+0	+0	+0
0	1	2	3	4	5	6	7	8	9	9	8	7	6	5	4	3	2	1

How are these examples alike? How are the examples different? After studying these examples for likenesses and differences, the child can discover that each addition example has one addend, a zero. The sum is the same as the

addend other than the zero. From studying the above examples, children are able to generalize that adding zero with another number doesn't change the non-zero number. This is called the identity element of addition.

The top row and the first column of inside Table 3 locates all of the basic addition facts utilizing the identity element of addition.

Table 3

Sums with One Element Being the Identity Element (0)

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

First Row

First Column

There are nineteen basic addition facts that can be related to the identity element of addition. Thus, remembering one rule, the identity element of addition, children can recall nineteen basic addition facts or about one-fifth of them.

Another set of basic addition facts may be related to the addition of one. Consider the examples below.

1	1	1	1	1	1	1	1	1	9	8	7	6	5	4	3	2
+1	+2	+3	+4	+5	+6	+7	+8	+9	+1	+1	+1	+1	+1	+1	+1	+1
2	3	4	5	6	7	8	9	10	10	9	8	7	6	5	4	3

How are all of these examples alike? How are all of these examples different? One of the addends in each example is one. The sum is one greater than the other addend. Children can generalize that adding one with a number gives the next number. These examples can be located in the second row and the second column of the addition table (Table 4).

Table 4

Sums With One Element Being 1

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Second Row

Second Column

Note that there are seventeen basic addition facts that can be related to this idea. If children can relate counting to these facts, then they will have these seventeen basic addition facts memorized.

There are nineteen basic addition facts that can be identified with the identity element of addition and seventeen facts can be related to counting. Thus, thirty-six basic addition facts, or more than one-third of the basic addition facts have been memorized by understanding and remembering two rules.

Let us now examine another subset of the basic addition facts. Children should have had experiences leading to this subset of addition facts. For example walking on a number line and saying the numbers upon which either the right or the left foot steps. (See Figure 2)

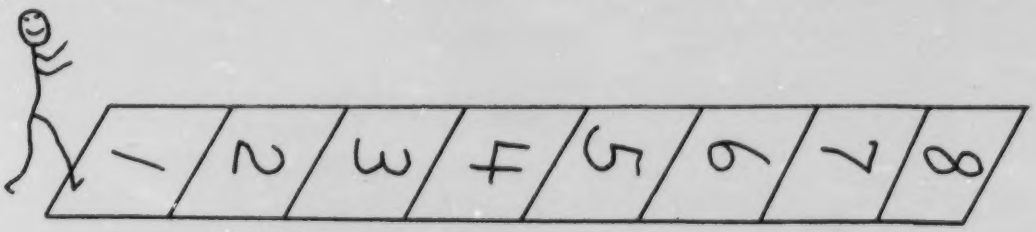


Figure 2

Walking on a Number Line

After this type of activity, children should be asked to count by two's. One must remember to tell the child with which number to begin counting. When counting by two's, 1, 3, 5, 7, 9, 11, 13, ... is just as valid as 2, 4, 6, 8, 10, 12, ... In fact, they should be able to begin at any number.

This set of fifteen basic addition facts would be related to the concept of twoness.

2	3	4	5	6	7	8	9	2	2	2	2	2	2	2	2
+2	+2	+2	+2	+2	+2	+2	+2	+9	+8	+7	+6	+5	+4	+3	+2
4	5	6	7	8	9	10	11	11	10	9	8	7	6	5	4

Note that this set of basic addition examples is located in the addition table in the third column and the third row (Table 5).

Table 5

Sums With One Element Being 2

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Third Row

Third Column

Children should be encouraged to discover that the sums can be derived by counting by two.

Placing these fifteen addition examples with the thirty-six addition facts already developed, we now have a total of fifty-one basic addition facts developed and memorized. The children only need to remember three basic rules or ideas and yet they have fifty-one basic addition facts memorized. Is it easier to remember fifty-one isolated facts or is it easier to remember three rules and relate the fifty-one examples to the rules? As children grow, develop, discover, and generalize; each class of examples becomes more difficult to generalize a rule.

The next set of addition facts I classify as the doubles. There are seven more of these examples and they are located in the addition table on the diagonal (See Table 6). These examples are:

3	4	5	6	7	8	9
+3	+4	+5	+6	+7	+8	+9
6	8	10	12	14	16	18

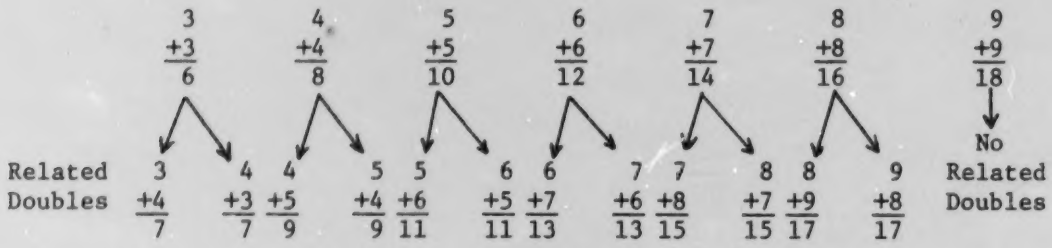
Table 6

Sums of Doubles

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Children have less difficulty memorizing this set of addition facts than many of the other addition facts. Since these seem to be easily memorized, we use this as a base for the set of basic addition facts which I call the "related doubles." The sum for each related doubles is one more than the sum for the double. Since 3 then 3 and 4. The twelve related doubles are:

$\frac{+3}{6}$ $\frac{+4}{7}$ $\frac{+3}{7}$



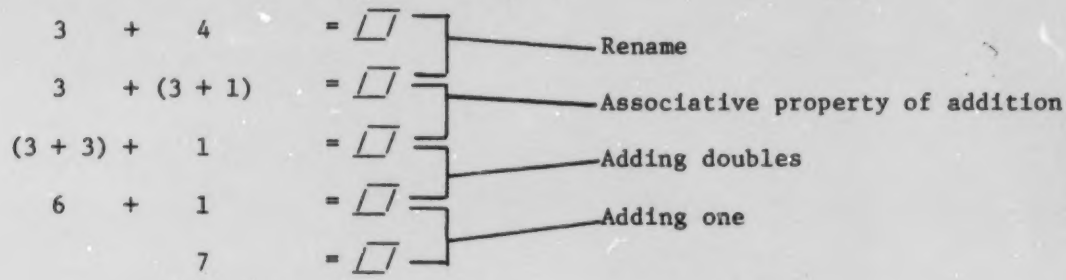
Note that there are twelve related doubles and they are located in Table 7 on two diagonals.

Table 7

Sums of Related Doubles

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Children may use the doubles, associative property of addition and adding one to a number as a technique of recognizing the sum. For example:



The seven doubles and twelve related doubles. along with the fifty-one facts already developed makes a total of seventy basic addition facts.

Next, since children should be developing an understanding of place value, they can bring together the concept of addition and the concept of place value for the set of basic addition facts which we will call "facts related to tenness." There are ten of these facts.

3	4	5	6	7	9	9	9	9	9
+9	+9	+9	+9	+9	+7	+6	+5	+4	+3
12	13	14	15	16	16	15	14	13	12

Table 8

Sums Related to Tenness

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

By studying these examples, children should generalize it takes one with nine to make a set of ten. The set of ten is placed in the tens place, leaving one less than the number being added in the ones place. For example, 9 + 5 = //, one from the five is placed with the nine to make ten. The example becomes 10 + 4 or fourteen. With a good understanding of place value, children have little difficulty discovering and applying this generalization of tenness.

Placing these ten examples with the seventy already developed, we have a total of eighty basic addition facts. That means there are twenty basic addition facts yet to be memorized. The commutative property of addition will reduce the number of basic addition facts to learn to ten. These are difficult examples for children to learn because there is no unifying pattern. (See Table 9.) Each child is encouraged to search for ideas that will provide individual help. The last set of facts is:

5	6	7	8	6	7	8	7	8	8
+3	+3	+3	+3	+4	+4	+4	+5	+5	+6
8	9	10	11	10	11	12	12	13	14

and the commutative property provides the other ten.

3	3	3	3	4	4	4	5	5	6
+5	+6	+7	+8	+6	+7	+8	+7	+8	+8
8	9	10	11	10	11	12	12	13	14

Table 9

The Remaining "Difficult Sums"

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Sometimes the comment is heard that if the children understand the commutative property of addition that this will cut the number of basic addition facts in half. This is not a true statement because the facts we have classified as doubles cannot be commuted. Forty-five of the basic addition facts can be commuted and ten of the basic facts cannot be commuted. The child still must memorize fifty-five of the basic addition facts and forty-five of the fifty-five basic addition facts can be commuted.

In working with young children and the basic addition facts, it seems advisable to develop the basic addition facts in families of numbers and memorize the facts according to structures. If a child is memorizing the basic addition facts by using structures, he is not learning the facts as specifics. Instead of memorizing one hundred unrelated facts, he is memorizing structures and then placing addition examples into the structure.

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PART II

DIAGNOSTIC TESTING

Chapter 6

ANALYSIS OF SOME AVAILABLE DIAGNOSTIC INSTRUMENTS IN MATHEMATICS

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University of Kansas
and
Charles F. Jenkins
Kansas City Missouri Public Schools

INTRODUCTION

The following eighteen tests were selected as a result of listing tests that appeared on a review of literature on remediation in mathematics. The tests selected are only a few of those available. However, they were those tests being utilized with some frequency in the research literature.

It should be noted that those tests actually used in classroom practice might differ significantly from those included here. Perhaps the reader can best judge the usefulness of this review by looking at those tests with which they are familiar.

The information for this report has been derived from the tests themselves and from conversations with teachers who used some of the tests. Thus, opinion must be separated from factual information as the reader digests the material.

REVIEWS OF TESTS

"The Adston Diagnostic Instruments (Elementary School Mathematics: Whole Numbers)," by Sam Adams and Leslie Ellis, 1971.
Published by Adston Educational Enterprises, Inc.
P.O. Box 18430-B, University Statio
Baton Rouge, Louisiana 70803

The Adston Diagnostic Instruments (ADI) have been designed for the purpose of providing the teacher with guidelines for remedial instruction in the area of addition, subtraction, multiplication and division of whole numbers. No specific age levels are indicated by the author, but the ADI can be used for all levels of elementary school mathematics where the content is applicable. The ADI determine what skills the student is deficient in, and provide evidence as to what additional skills are needed in order to master the next operation. From such evidence, remediation approaches may be developed. There is no specific time limit. They can be repeated several times in a brief span of time. The only concern is that the instruments be used in whatever manner seems to be indicated in order to locate the difficulties of a particular student.

In the developmental states, the ADI was administered to 100 upper elementary school students from several schools in Livingston Parish, Louisiana as a part of the refinement procedures for the test. The test is not normed.

The ADI is considered a valid test in as much as it does test what it purports to test, as the authors point out.

The reliability or internal consistency of the ADI was determined by using the Kuder-Richardson Formula 20. The reliability coefficients on specific operations are listed below:

Addition Diagnostic,	r = .88
Subtraction Diagnostic,	r = .92
Multiplication Diagnostic I,	r = .93
Multiplication Diagnostic II,	r = .93
Division Diagnostic I,	r = .94
Division Diagnostic II,	r = .92

Since the range of the reliability coefficients was from .88 to .94, this can be considered to be strong evidence for the consistency of the test-retest score measures of the ADI, and the reliability can be said to be good.

The ADI appears to be a good test for the basic operations and the basic facts of multiplication, addition, subtraction and division. The manual is well written and the comparison between achievement tests and diagnostic tests is helpful in understanding the test.

"The Prescriptive Mathematics Inventory" by John Gressel, 1972
Published by CTB/McGraw-Hill Company
Del Monte Research Park
Monterey, California 93940

The Prescriptive Mathematics Test (PMI) is an effective educational tool for individualizing instruction through applying specific diagnosis and corresponding relevant treatment. It is intended for use as an evaluation measure before and after instruction. Used before instruction, the PMI assesses each student's mathematical knowledge and diagnoses his specific weaknesses. When used after a period of instruction, the PMI evaluates the student's progress toward the mastery of certain objectives. The test-teach and then re-test cycle is repeated throughout the application of the program. The PMI is very useful in developing either remedial or enrichment activities for continuous programming for the stated objectives.

The tests that are available are coded as follows: (1) the Orange PMI test ranges from grade 4 to 5; (2) the Aqua PMI test ranges from grades 5 to 6; (3) the Purple PMI test ranges from grades 6 to 7; and (4) the Level C PMI test ranges from grades 7 to 8. Criterion is generally 90% of the items on one of the tests before the next color test is used.

After the pretest is given, a recognition of deficiencies in certain of the stated objectives is made. Then these specific objectives are taught through concepts to remediate the deficiencies. Then a posttest is given to determine the result of the remedial teaching. This test-teach-test cycle is repeated throughout the program.

The objectives measured by the PMI have been keyed to a number of widely used mathematics textbooks series in use. Based on these textbooks, an Individual Study Guide will be produced for each student. After the individual diagnosis is made and the Individual Study Guide is formed, the teacher can plan instruction and study assignments with a student to meet his particular needs and interests.

The PMI is intended to determine the individual student's knowledge of basic mathematics. It is not intended to classify him or compare him with other students or with some norm group.

The PMI takes a total time of 3 to 3½ hours to administer. It is suggested to schedule several 40 minute testing sessions over a period as long as a week.

The PMI has no norms. Furthermore, no validity or reliability data was given.

The general format of the PMI is a positive feature. The key to objectives needs to be updated to include current text materials. In the teaching cycle, the stated objective should be kept paramount in the mind of the student, so he may see the related connection between the study activities and the objective.

Repeated use of the tests to determine criterion levels tends to invalidate the specific test as a measure of mastery or criterion levels.

"The Watson Diagnostic Mathematics Test" by G. M. Watson, 1973
Published by The Book Society of Canada, Limited
4386 Sheppard Avenue E
P.O. Box 200
Agincourt, Ontario M1S 3B6

The purpose of this test is to assist the teacher in diagnosing the areas of difficulty that a student is having, to determine the individual ability of the student, and to provide a corresponding standard of measurement.

The Watson Diagnostic Mathematics Test (WDMT) provides the teacher with a tool which enables her to identify immediately what skills a pupil has not mastered, so that those skills can be retaught and relearned.

The WDMT has tests published for the primary (1, 2 and 3 grades), junior (4, 5 and 6 grades), and the intermediate levels (7, 8, 9 and 10 grades). For each level, equivalent forms of A, B and C are available.

These different forms may be given at two or three monthly intervals so that progress can be noted and grouping for reteaching organized.

The primary tests cover basic concepts, number systems, counting, simple addition, subtraction, and multiplication, up to the six times tables; the junior tests cover all addition, subtraction, multiplication, division and the simpler processes of fractions, decimals and percentages; while the intermediate tests are a more complex version of the junior form-level.

In the primary form, the test times are divided into 3 years or grade levels--part I covering the work of the first year, part II that of the second year, and part III that of the third year. The primary forms are organized horizontally, so that the pupil may work from easy to slightly more difficult materials and items, to avoid frustration.

The junior and intermediate forms are constructed vertically for each process of addition, subtraction, multiplication and division.

The WDMT has come about as the result of eighteen years of testing and experimentation in the Hamilton Public Schools, using their own course of study. The first edition was used for fifteen years as both a diagnostic and terminal test. The items of the second edition became the basis for the third and fourth editions, for all the items were the same, only being arranged in a different order. Norms for the second and third editions were established by using 300 pupils in four different schools. The fourth edition was normed on 3000 pupils in nine different schools in different socio-economic areas (the norms are applicable only when interpreted at the close of a school year as a terminal test).

All tests of the WDMT were designed to be completed in one half hour.

The author illustrates evidence of validity and reliability through the use of various graphs. No specific statistical correlation or score was given for the validity or reliability of the tests.

Generally, the WDMT are solid tests that can be used profitably by a teacher to help determine the strengths and weaknesses of mathematics students. Since there are few word or thought problems in the test, the test clearly is not measuring reading skills. Division symbolization is different and would need to be explained to American students.

"The Key Math Diagnostic Arithmetic Test" by Austin J. Connelly, William Nachtman, and E. M. Pritchett, 1971.
Published by American Guidance Service, Inc.
Publisher's Building
Circle Pines, Minnesota 55014

The Key Math Diagnostic Arithmetic Test (KMDAT) is an individually administered test which has been designed to determine the diagnostic assessment of a child's basic skills in mathematics. By identifying the individual's strengths and weaknesses, a program of remediation may be

developed and appropriate individualized instruction may begin to be taught to the child.

The KMDAT is generally used from preschool through grade six, although there is no upper limit for individual clinical and remedial use.

The KMDAT is organized into three subtests which are represented by (1) content (numeration, fractions and geometry and symbols); (2) operations (involving addition, subtraction, multiplication, division, mental computation, and numerical reasoning); and (3) applications (involving word problems, missing elements, money, measurement and time).

An early pilot test consisted of approximately 1400 educable mentally retarded youngsters in some 20 states in which the items were screened. The final part of the test items was administered to 320 children in Columbia, Missouri. The Rasch and Wright procedures were used to evaluate item performance across the total range of the test to determine if the item would "fit the model."

The calibration sample consisted of 951 students from the elementary through the junior high grade levels in four states. The Rasch and Wright procedure was used to make item analysis interpretations and to correlate them with the subtest and total test scores.

The KMDAT has grade norms drawn from 1222 subjects of K through 7 grades taken from 42 schools in 8 different states for those who feel that they need norm scores. But since the KMDAT is considered a diagnostic tool, the main emphasis placed on the test interpretation should be the interpretation of strengths and weaknesses as represented by the subjects performance in subtests and on individual items. It is from this data that remedial programs can be generated.

Reliability measures (determining the stability of scores obtained by a given individual when successive measures of the same or similar form of the test are taken) showed the KMDAT to range from .94 to .97 across different grade levels. This is very good reliability.

The authors have indicated that the KMDAT does have content validity (indicating that the test does cover an adequate sample of the specific content under consideration), face validity (which refers to not what a test actually measures, but to what it appears to measure, based on subjective evaluation), and concurrent validity (the comparison of the test with some other comparable criterion of known validity) by comparing the KMDAT with the Iowa Basic Skills Arithmetic portion to obtain a concurrent validity of .69 correlation with the reasoning measure of a .38 with the full scale Iowa Arithmetic score.

The Key Math Diagnostic Arithmetic Test has colorful eye appeal, is easy to administer, and can be scored quickly. The information that can be secured from the KMDAT is valuable in determining the strengths and weaknesses of students, so as to allow the teacher to plan a remedial program to correct the deficiencies that the student may be having. The

Diagnostic Profile Record is a splendid graphic medium to observe visually and compare the student's strengths and weaknesses.

It would be helpful to have more definitive data as to the reliability and validity of the test. In time, as the KMDAT accumulates field evidence, reliability and validity data should be available.

"The Stanford Diagnostic Arithmetic Test," by Leslie S. Beatty, Richard Madden, and Eric F. Gardner, 1966.
Published by Harcourt Brace Jovanovich
757 Third Avenue
New York, New York 10017

The Stanford Diagnostic Arithmetic Test (SDAT) is a test designed to identify the weaknesses and strengths of children in the area of arithmetic. The test content and suggestions for use of results are based on the general assumption that in order to improve pupil competence in the various phases of arithmetic, it is first necessary to find out what specific learning problems the pupils have and then begin remediation on these difficulties.

Two equivalent forms of the SDAT Level I (prepared for the latter part of Grade 2 through the middle of Grade 4) are named Form X and Form W. Also, two equivalent forms of the SDAT Level II (prepared for the latter part of Grade 4 through the middle of Grade 8) are named Form X and Form W. These two equivalent forms have been prepared to allow equi-test measurement procedures in administration. Both levels of test cover concepts of number and numerals, computation on all operations, fractions, decimals, percent, and the basic number facts.

The SDAT can be administered within a few days or over a longer period of time as the teacher feels a need for information about pupil performance in the various areas tested.

The SDAT has a Pupil Profile Chart, a Class Analysis Chart, and a Class Record for easy visual comparison of the different subtest scores.

In the original development of the SDAT, the items selected were analyzed using 15,000 pupils from seven school systems in five separate states. The standardization program was performed in October, 1965 using 8000 cases taken from four different school systems. From the total groups of pupils tested, samples were selected for the development of norms. The tests were normed from pupils from average median family income and average median years of school completed by the adult population. The school systems used were in a traditional phase in going from a traditional to a modern mathematics program. The tests are normed on stanine scores. The original standardization took place on Form X while Form W was normed by equating scores from Form X on a corresponding subtest equi-percentile procedure.

Reliability data was secured for the SDAT using a corrected split-half (odd-even) procedure obtaining reliability coefficients and standard errors of measurements which resulted in superior reliability data. Most subtests

ranged in the .80's to the .90's, which is considered good. The manual has two tables listing the subtests and their reliability coefficients and standard errors of measurement for the Grades 5, 6, 7 and 8.

The validity data of the SDAT has been interpreted on those specific comparable content parts of the Stanford Achievement Test using the pupils in the Form W norm samples. Specific subtests seem to indicate about average concurrent validity when based on the Stanford Achievement Test comparison. The authors indicate that the validity check is weak and needs a more formal analysis.

The SDAT offers consistent and reliable data on the deficiencies of children who are having difficulties in arithmetic. The ease of presentation (either group or individual), the ease of scoring, and the thoroughness of the content analysis makes the test very valuable to the classroom teacher. The test can be given by the classroom teacher and enough specific information can be obtained about the difficulties that the pupil is having in arithmetic so that an immediate remedial program can be planned to help the pupil.

The SDAT could be improved with color print designs displayed throughout the problem so as to generate more attractive interest in taking the tests. The total test does require a large investment of time, which may not be required because many of the deficiencies in content areas could be determined through the use of informal interviews with pupils.

"The Project for the Mathematical Development of Children" by Tom Denmark, 1976.
Florida State University
Tallahassee, Florida 32306

The Project for the Mathematical Development of Children (PMDC), according to the examiner's manual, is an individually administered test designed to assess a student's attainment of mathematical concepts and skills. All directions are given orally and/or by demonstrating the solution to a problem by using various manipulative aids. The test items are organized from easy to hard through Grade Level 1 and through Grade Level 2.

The student observes the test items from an easel notebook display and then responds orally to each item. The examiner records the response on the Response Form. Manipulative materials include beans, duck pictures, cups, stars, circles, and a square. The total results are written on a student profile sheet used to summarize the student's responses on the content scales.

The tests are printed for Grade 1 and Grade 2 levels. There are no norms on the tests.

The PMDC (Grade 1) was first administered to 197 first grade students, 97 boys and 100 girls, during the first three weeks of September, 1975. The students came from two schools located in Tallahassee, Florida and

three schools in Athens, Georgia. The schools selected had a large variance in race, parental education, family income, and parental occupation. The PMDC (Grade 1) covers the areas of counting, set equivalence, ordering of numbers, addition and subtraction, and class inclusion.

The reliability analysis is reported using Cronbach's alpha and provide a significant range of reliability from .64 to .93, indicating reasonable to good reliability on each content area. The Spearman-Brown split halves formula was used to compute total test reliability and yielded a reliability of 0.95, which is quite good.

The Key Math and Otis-Lennon tests were given to all students, resulting in the following correlations with the PMDC:

PMDC and Key Math.....0.79
PMDC and Otis-Lennon.....0.72
Key Math and Otis-Lennon.....0.75

A moderate degree of concurrent validity seems apparent.

The PMDC (Grade 2) was administered to 158 entering second grade students, 81 boys and 77 girls, during the first three weeks of September 1975. The students came from two schools in Tallahassee, Florida and three schools in Athens, Georgia. The content areas of the PMDC (Grade 2) includes elementary counting, advanced counting, patterns, place value, equivalent number names, addition, subtraction, missing addends, ordering numbers, and class inclusion.

Cronbach's alpha reliability coefficients ranged from 0.47 to 0.95 on the different content areas. The Spearman-Brown split halves formula was used to complete a total test reliability of 0.93.

Concurrent validity between the PMDC, Key Math, and the Otis-Lennon tests are listed below:

PMDC and Key Math.....0.82
PMDC and Otis-Lennon.....0.68
Key Math and Otis-Lennon.....0.78

This appears to be a moderate degree of concurrent validity.

The PMDC is a good determiner of mathematical skills for children in Grades 1 and 2. The use of manipulative aids is good for this age child. Children who show deficiencies in various essential areas can be given remediation to correct their defects.

Since the test requires only an oral response from the student, and is not a paper-pencil test, the child should be able to respond more freely and do better on the test.

"Project Torque: A New Approach to the Assessment of Children's Mathematical Competence," 1976.

Education Development Center, Inc.,
55 Chapel Street
Newton, Massachusetts 02160

Project Torque (PT) is a series of tests which are structured for the upper elementary school level (the lower elementary tests are being developed at this time). The test items come from the source of games and activities. The tests evaluate measurement, computation, estimation, mapping and scaling, graphs, modeling parts of the real world using numbers.

In these games of PT, a child's ability to perform depends directly upon his/her competence at a specific set of mathematical skills. The activity or game that the child performs is called a "Validating Instrument." Shorter test forms that take the form of games and activities are called "Diagnostic Achievement Tests." Direct observation of the student may not be necessary since the test (puzzle or game) can be scored systematically based upon the raw score earned by the student.

Since there is a minimum of reading and writing of words, the PT is not considered to measure reading.

The sequential PT model is essentially (1) the development of a game illustrating a mathematical concept or skill, (2) a clinical testing of the game to form the Validating Instrument, (3) the development of a shorter written record called a Diagnostic Achievement Test which is tried out with many students, (4) giving both the Validating Instrument and the Diagnostic Achievement Instrument both yield results that are criterioned referenced. Each student's profile will reflect the criterion that he or she can reach.

Project Torque follows these guidelines in the evaluation of students: (1) the tests should be diagnostic, identifying the strengths and weaknesses of children, (2) the tests should be sensitive to each unique child and should be tests that are individualized and culture free, (3) the tests should get the child involved in doing his best in a non-threatening atmosphere, (4) the tests should be criterioned referenced so as to assess performance on a specific task, and (5) the tests should be validated from actual observations of behavior in various tasks, and finally, (6) the tests should be easy to administer.

The concept of using games to test the deficiencies of children is an interesting approach. The important thing that needs to be considered is, does the game have enough item identity or specificity to allow the teacher to really know that the child does have a deficiency in the area being tested? The latitude of the game is important. The example of the Pinball Wizard jumps from add to subtract, to multiply and divide, etc. Would it be better to construct an activity or game only on addition, or only on subtraction, etc., and then to analyze the operation specifically to determine the precise difficulty the child is having instead of presenting multiple operations with no specificity? It might be better to construct

only one activity or game on one specific operation so to be able to analyze pupil performance on that specific operation more precisely.

"The Arithmetic Concept Screening Test," by Herbert Goldstein, 1969.
Curriculum Research and Development Center in Mental Retardation
Department of Special Education
Ferkauf Graduate School of Humanities and Social Sciences
Yeshiva University
55 Fifth Avenue
New York, New York 10003

The Arithmetic Concept Screening Test (ACST) is designed for low mental age, primary, or intermediate retarded children. Only minimal verbal comprehension, reading, and writing skills are involved in responding to the test. The test items are presented pictorially and the children are simply asked to mark an "x" on the correct answer space in their test booklets. There are two equivalent forms of the test--the G Form and the H Form.

The primary function of the ACST is to place and screen the children at the proper level of difficulty and to pinpoint missing arithmetical concepts and skills. The ACST has been designed essentially for student placement, evaluation, educational measurement and evaluation research.

The test is divided into six sections, each with its own test booklet. The problems presented are judged to be homogeneous in terms of their level of difficulty. The six areas are the Readiness Level, Levels I, II, III, IV, and V. In the Readiness Level form and size discrimination, one-to-one correspondence and basic concepts are covered; Levels I and II cover rational counting, visual clusters and grouping, and rote counting; Level III involves reversibility and addend rules; Level IV covers rational counting, ordinal numbers, basic addition and subtraction facts, and multiplication and division readiness; Level V covers number name recognition, written symbols, place value, money concepts, addition, subtraction, multiplication and division.

The Kuder-Richardson 20 test of internal consistency was used to determine the reliability of the ACST. The reliability of each level was: Readiness Level = 0.74; Levels I and II = 0.82; Level III = 0.90; Level IV = 0.89; and Level V = 0.95. There are considered to be good reliability coefficients.

Construct validity tests were made on the ACST which resulted in moderate to high correlation.

Concurrent validity was studied using the Standford Binet and the WISC IQ and MA estimate scores with the ACST which resulted in moderate to high correlation.

A test dealing with arithmetic concepts for the retarded population certainly is needed. The ACST appears to be an adequate test and contextually, is reasonably thorough for that population.

"The Basic Educational Skills Inventory-Math Levels A and B," by Gary Adamson, Morris Shrage, and Glen Van Etten, 1972.
Marketed by Select Ed.--Incorporated
Winch and Associates, Box 1185
Torrance, California 90505

The Basic Educational Skills Inventory (BESI) originated at the Educational Modulation Center in Olatha, Kansas under a federal grant. It was field tested by Gary Adamson, Morris Shrago, and Glen Van Etten and was revised and published successfully as an experimental edition of a diagnostic tool in math.

The BESI is used extensively in special education diagnostic batteries and is considered to be a very good informal diagnostic tool in the area of math.

The BESI covers addition, subtraction, multiplication and division problems and facts; fractional parts; renaming of fractions; addition, subtraction, multiplication and division of fractions; addition, subtraction, multiplication and division of decimals; decimal-fraction-percent transformations; time; and money.

Pretesting, remediation, and then posttesting result in an initial score, a final score, and a positive or negative gain or loss score.

A scoring booklet provides specific activity descriptions with descriptors and spaces for remedial methods and materials. Response booklets are provided for Levels A and B in which the student responds to all of the activities.

The formal and design of the BESI, along with its mathematics content, seems to be a good, informal diagnostic procedure in mathematical evaluation.

"Orleans-Hanna Algebra Prognosis Test," by Joseph B. Orleans and Gerald S. Hanna, 1968.
Published by Test Department
Harcourt Brace and World, Inc.
757 Third Avenue
New York, New York 10017

The Orleans-Hanna Algebra Prognosis Test (OHAPT) may be used as an instructional aid for planning lessons and assignments that will meet the needs of students of different abilities. The primary use of the OHAPT is for identifying, before instruction, those students who may be expected to achieve success and those who may be expected to encounter difficulties in algebra courses. The OHAPT combines in a simple and economic manner, a

variety of predictive data that yield important information about individual students. This data can be used to assist the student to effective remedial instruction in algebra. Raw score interpretations and item analysis responses will help in understanding deficit areas so that remedial teaching may take place.

The format of the test follows first a fundamental learning sequence that the student studies, and this is followed by an immediate testing situation built on the content of the material studied. The test takes about 55 to 60 minutes to administer, and can be used for grades 8 through 11.

The OHAPT was standardized on 5,109 students, representing a sample of the national population of public secondary school students who were defined on the basis of geographic location, size of the school district, and the socio-economic status of the community. Twenty-five separate school systems from across the nation were enrolled in the standardization.

The test has content and construct validity. The predictive validity (judging the effectiveness with which the test predicts success in learning algebra) ranged from between 0.72 to 0.84, which indicates the OHAPT is performing quite well as a predictor of success in learning algebra.

The reliability of the test (the extent to which the test yields consistent measures) was 0.95, which is considered a strong reliability test.

Especially desirable is the restricted content learning experiences being followed immediately by the performance test. This idea is fundamental in determining the ability of the student in algebra. One wonders though, if the pretest experiences and the learning of algebra is thorough enough to allow the student enough mastery to perform reasonably well on the test.

"Orleans-Hanna Geometry Prognosis Test," by Joseph B. Orleans and Gerald S. Hanna, 1968.
Published by Test Department
Harcourt Brace and World, Inc.
757 Third Avenue
New York, New York 10017

The Orleans-Hanna Geometry Prognosis Test (OHGPT) may be used as an instructional aid for planning lessons and assignments that are individualized and that meet the needs of students of different abilities.

The OHGPT combines in an objective, simple, and economic manner, a variety of predictive data that yield important information about individual students. Taken in conjunction with other reliable information, the test results can be used for counseling students and classifying them for specific geometry instruction.

The OHGPT takes about 55 to 60 minutes to administer, depending on the type of answer sheet being used. The test should be completed in one setting. The grade range for the OHGPT is from 9 to 11 grades.

The OHGPT was standardized on 3,525 secondary school students, from a population defined on the basis of geographic location, size of the school district, and socio-economic status of the community.

The test has content and construct validity. The predictive validity ranges from 0.61 to 0.69, which is evidence the test is performing well as a predictor of success in learning geometry. The reliability of the test averaged 0.86, which is considered strong consistency in measuring repeat items on the test.

The concept of preteaching and immediate testing following the pre-learning experiences is excellent. When the test is given immediately following the learning of the facts being tested, the immediate recall by the student will help to determine his basic capacity to do well in geometry. When used diagnostically, this test can give evidence of weaknesses that need remedial attention. When proper instructional materials are used to correct the weaknesses of the student, apparent defects can be corrected.

"Diagnosis: An Instructional Aid Program," by Science Research Associates, Inc., A Subsidiary of IBM, 1975.
Published by Science Research Associates
259 East Erie Street
Chicago, Illinois 60611

Diagnosis: An Instructional Aid Program (DAIAP) is a new approach to diagnosing students' specific strengths and weaknesses in mathematics. The program includes prescriptive activities to correct the weaknesses of students.

The teacher is able to see precisely which skills an individual student has not mastered, and can direct the student to a number of texts and/or learning kits for related study.

The DAIAP is built around probes or short criterion-referenced diagnostic tests. Each probe measures specific learning objectives and is designed to pinpoint specific skill weaknesses. The learning objectives are coded to an Index Code for use with the Prescription Guide. The mathematics labs have 30 consumable copies for the students to use for each probe. The Prescription Guide correlates probes to pages in a number of basal textbooks and SRA supplemental materials. The diagnosis labs seem to work better when several references and textbooks are used. The Prescription Guide lets the student know exactly where he or she will find the help needed.

The program contains a Class Progress Wall Chart, on which the probes that each student should take are marked. These are checked off as the student completes them.

The DAIAP is structured in two levels, Mathematics Level A (grades 1-), and Level B (grades 4-6). Level A contains 24 probes covering basic computation, set theory, numerals, operations and problem solving, measurement, and geometry. Level B contains 32 probes, covering whole-number computation, fractional-number computation, decimal-number computation, numeration theory, operations, problem solving, measurement, and geometry.

Two different survey tests are available for pretest and posttest use. Form X can be used for a pretest. Form Y can be used as a posttest after the probes, prescriptions, and remediation have been completed by the student, to determine if adequate progress and remediation have been accomplished.

There is no reliability or validity information given for the DAIAP.

The DAIAP is an excellent approach to diagnostic-prescriptive teaching in the area of mathematics. The kit is rather expensive.

Since motivation is so important in remedial mathematics, one wonders if a student, when left to independent study in working through the prescriptive materials, would be able to make the needed progress by self-motivation only.

"Diagnostic Tests and Self-Helps in Arithmetic," by Leo J. Brueckner, 1975.

Published by CTB/McGraw-Hill
Del Monte Research Park
Monterey, California 93940

The Diagnostic Tests and Self-Helps in Arithmetic (DTSA) have been developed to enable teachers to make a developmental diagnosis of the basic skills in all operations with whole numbers, common fractions, decimal fractions, percents, and measures. The DTSA determine specific weaknesses of students, so that corrective and remedial work can be processed to remedy any deficiencies that students may have. The Self-Helps, which appear on the reverse side of each test sheet, are geared to provide the needed remediation for the student.

The DTSA consists of twenty-seven tests, where the first four are screening tests which are used to determine quickly the student's general areas of deficiency in whole numbers, common fractions, decimals, denominate numbers, and percents. The other twenty-three tests are specific diagnostic tests which are used to determine precisely where the student's skills breakdown in the basic facts, the fundamental operations on whole numbers, common fractions, decimal fractions, percents and operations with measures.

The DTSA can be given individually as well as to groups. The DTSA can be given at any grade level where the content is appropriate.

In the fall and spring of the year, the student can be given the DTSA as a systematic review of all basic operations.

The DTSA is not standardized or normed, for it is a diagnostic test that helps the teacher to identify specific areas of weaknesses that the student is having.

Cross-referenced tables are provided for teachers who use the California Arithmetic Tests to help determine which of the DTSA subtests the student will profit most from taking.

As a measure of the success of the remedial work, a Diagnostic Test may be repeated at the end of the remedial work or at any other time it seems advisable. At the upper grade levels, the Screening Test may also be repeated as often as needed.

Class Record Sheets are available to record total class scores for comparison. Individual Record Sheets are available to record each student's responses on the Diagnostic Tests.

The DTSA can be coordinated effectively with any series of arithmetic textbooks.

The author does not give any specific information or data on the reliability or validity of the DTSA, but he does assure us that the DTSA is a reliable and valid tool.

The size of print is too small, particularly so, with the fraction test. Some of the serial operations should be varied to allow more evident mastery of the operation. The Self-Help activities could be presented in a more interesting fashion, and there really should be more activities to assure adequate remediation.

"The Pattern Recognition Skills Inventory," by Les Sternberg, 1974.
Published by Hubbard Scientific Company
2855 Shermer Road
Northbrook, Illinois 60062

The Pattern Recognition Skills Inventory (PRSI) is a tool which is designed to help educators determine the readiness levels of students in language and mathematics. The PRSI has appropriate curriculum approaches which can actually teach and improve recognition of patterns in pupils. The PRSI can be used as a measuring tool and a diagnostic-remediation process tool. It is effective in remediation of pattern recognition difficulties.

In the PRSI, the student structures abilities to identify, discriminate, and match several topological items by color, picture, and pattern. Skills are also structured to discriminate sequential order of items. A pattern sequence (involving a stimulus dimension) would follow color, specific name, specific label and organization by category. The child is expected to pattern in one of these ways.

Patterning by color sequence would involve recognizing square color plates in pattern. Specific name patterning would involve recognizing

identical picture sequences by names that are associated with the pictures, as in dog/cat/pear, etc.

Specific-label patterning is the skill of recognizing similar objects or animals in different positions, as in a dog running or lying down.

Categorical patterning is the organizational skill of grouping objects that are not identical but, due to some degree of similarity, belong to the same category.

The PRSI may be given individually or to the entire class by the examiner. The examiner presents a model to the child, which is followed by four multiple choice responses, from which the child may choose the correct patterning as interpreted from the given array.

Pattern tasks are structured to (1) repeat and duplicate the original task--"original learning," (2) interpret patterns that are opposite the model presented--"the reverse shift," (3) recognize that the model and the response has the same basic pattern--"intradimensional pattern," and finally, (4) interpret tasks having the same patterns as the model but differing in some extra attribute such as color, shape, etc.--"extradimensional shift."

The PRSI is arrayed in four sets. Each set contains 24 patterned tasks.

In one-to-one testing, the examiner records the subjects responses on a response evaluation sheet. Group patterns are recorded on other evaluation sheets.

There is an answer key on the reverse side of each model which gives the task number, type of task, the dimension, and the pattern involved.

In recording, a (+) is used for correct answers and a (-) is used for incorrect answers. Each pattern will have a total score of (+) responses for total tabulation, thereby determining the child's ability to recognize the pattern involved. Six tasks out of 24 is no true pattern recognition ability, where as 10 tasks correctly scored out of 24 shows significant pattern recognition.

Since the ability to recognize patterns can be viewed as a prerequisite to certain types of higher mathematical or language functioning, then, students who score low on the PRSI would need remediation and training in the patterning tasks. The author indicates process methods and outlines approaches to teaching pattern recognition.

The author believes there is an hierarchy of skills in pattern recognition which follow the following sequence from the least complex to the most complex: (1) original learning; (2) the reverse shift; (3) the intradimensional relation; and finally (4) the extradimensional shift.

There is no reliability or validity data presented in the manual.

One may not agree with the basic thesis of the author about the hierarchical levels of pattern recognition and the need for remediation at these levels.

The PRSI format is good. The manual is well-written, with clear and logical interpretations of pattern recognition definitions.

The PRSI appears to be a useful tool for recognizing, diagnosing and remediating difficulties in pattern sequencing.

"The Diagnostic Fractions Tests," by E. W. Seville, 1966.
Published by The Australian Council for Educational Research
Fredrick Street
Hawthorn, Victoria, Australia 3122

The purpose of the Diagnostic Fraction Test (DFT) is to diagnose the particular errors which are met by pupils in dealing with fractions, with the intent of focusing attention on individual difficulties, and then helping the children to overcome them.

The DFT problems are structured and classified into certain types and have descriptive identities on the right side of each page describing the specific types of problems. The student proceeds from one type to the next in an easy going progression, going from easy to more difficult problems. Each classification is devoted to only one level of difficulty.

The DFT provides the teacher with a means of testing, from time to time, the efficiency of their teaching in the subject, and of diagnosing weaknesses, so that remedial action may be taken.

When the student's weaknesses have been pinpointed, thorough reteaching must be given. Every effort should be made to help the child understand the meaning of fractions. "Rules of thumb" alone are inadequate, for learning must be based upon thorough understanding.

Some remedial materials suggested by the author to help correct the difficulties that a child is having are "Easy Steps in Arithmetic-Fractions," and "Improve Your Arithmetic," as referenced in the manual.

The DFT may be used with individual children or with groups of children or with an entire classroom of children.

The data from the DFT help the teacher to become informed on the relative level of understanding of each child, the extent of individual differences, and the nature of specific difficulties at the individual or group level. Knowing this, the teacher can thereby be able to develop further remediation experiences covering fractions which would be adapted to the needs and abilities of the learners.

The DFT are published in one 8 page booklet, and contain four subset groups of fractional exercises classified as addition, subtraction, multiplication and division.

There are no time limits to giving the DFT.

There is no reliability or validity data given by the author.

The manual provides a clear description of procedures for the administration of the DFT and it is well-written.

These tests can be very effective tests in determining, diagnostically, the weaknesses of students. The strength of the remediation materials is very important in correcting deficits. Therefore, the materials should be motivating, interesting, and have simplicity and clarity.

"The Diagnostic Decimal Tests," by E. W. Seville, 1976.
Published by The Australian Council for Educational Research
Fredrick Street
Hawthorn, Victoria, Australia 3122

The Diagnostic Decimal Tests (DDT) provide a test which is clearly diagnostic in nature. The DDT purports to be able to diagnose a child's deficiencies in decimal fractions and then, to provide a constructive and remedial procedure to correct those specific difficulties.

The tests are arranged into three four-page booklets, covering Diagnostic Decimal Tests I (addition and subtraction), Diagnostic Decimal Tests II (simple multiplication and division), Diagnostic Decimal III (long multiplication and long division tests).

The tests can be given to individual students or to small groups of students, or to a whole class of students.

The format of the tests is such that problems are classified from easy to more difficult and are arranged by descriptive identities on the right side of each row of problems. This information can help the teacher to determine specific type-weaknesses, so that adequate exercises and remediation can be developed for the learner.

The DDT provides the teacher with a means of testing the efficiency of their teaching in the subject, and of diagnosing weaknesses so that remedial correction may be made.

After the DDT has been administered and the student's weaknesses have been pinpointed, thorough reteaching must be given. Every effort should be made to have the student understand the algorithms and conceptualizations that are involved in the decimal divide model.

With the information that the DDT provides, the teacher should become adequately informed on the relative level of understanding of each child, the extent of individual differences, and the nature of the specific difficulty at the individual or group level. Cognizant of this, the teacher would thereby be able to develop remediation experiences on decimals which would be adapted to the needs and abilities of the learners.

The authors of the DDT have given no reliability or validity data.

The descriptive format of each row of problems and the quick, easy methods of hand-grading the tests are desirable features. Since there are no time limits on the tests, a more accurate evaluation may be made concerning the deficiencies of learners who would be slow in responding to the tests. The authors indicate that the tests may be given at any age where the content of the test seems appropriate for the curriculum presented, which, therefore, makes the test a more flexible and useful instrument.

"The McGraw-Hill Basic Skill System-Mathematics Test," by Alton L. Raygor, 1970.
Published by CTB/McGraw Hill
Del Monte Research Park
Monterey, California 93940

The McGraw-Hill Basic Skills System-Mathematics (MHBSSM) is a flexible and well-integrated evaluation system which provides a group of tests and associated instructional materials which are designed to provide deficit-specific training in mathematics for students in grades 10 through 13. The diagnostic-remediation cycle of evaluation and then instruction are built into the system.

A major purpose of the MHBSSM is to provide adequate and accurate skills for college bound students.

The MHBSSM comes in six areas: Reading, Writing, Spelling, Vocabulary, Study Skills and Mathematics. Each of these areas is processed through four main evaluation stages; viz., (1) diagnosis, (2) prescription, (3) instruction (where self-instruction programs, workbooks, audiotapes, memory cards, etc. are used to correct deficit areas), and (4) evaluation.

There are two forms of the MHBSSM, Form A and Form B. One test can be used as a pretest and the other used as a posttest. Each form of the math test is divided into three parts; viz., Arithmetic, Elementary Algebra and Intermediate Algebra.

The test can be given within a class period of 42 minutes.

The test is designed to be diagnostic only in a general way. A low score implies a need for remedial work in math.

The test can be hand-scored or machine scored.

Three kinds of derived scores are available for each raw score; viz., percentile rank, standard score and stanine. The test is normed.

The test can be given to individual students or to groups of students.

Using a sample of N = 1320 for Form A and N = 1312 for Form B, the authors have provided data on the internal consistency of the MHBSSM test.

Using the Kuder-Richardson 20 formula, Form A was found to have a correlation of 0.94 (with a standard error of 3.11) and Form B had a correlation of 0.95 (with a standard error of measurement of 3.06). The MHBSSM has an adequate content validity, according to the authors, after they had used a point biserial correlation procedure. The MHBSSM test has not been correlated with a criterion validity score of any kind yet.

The MHBSSM is a useful test for determining weaknesses in 10 through 13 grade-level students who are college bound. The remediation materials that are recommended were not examined so their effectiveness and adequacy would need to be determined.

"Seville Diagnostic Number Tests," by E. W. Seville, 1966.
Published by The Australian Council for Educational Research
Fredrick Street
Hawthorn, Victoria, Australia 3122

The Seville Diagnostic Number Tests (SDNT) are tests which cover the four basic processes of addition, subtraction, multiplication and division, and are aimed at helping children overcome mistakes due to defective knowledge. of such basic knowledge as "carrying," adding 'seen' to 'thought-of' numbers, misunderstanding of handling blank spaces in the tens or hundreds column, adding zeros, addition of a number which produces a zero in their sum, etc. The tests in each process or operation are valuable at any stage of development after the children have been taught the process concerned.

The tests may be used with individual children who have been identified as retarded in arithmetic. They may also be used with an entire class as an integral part of the teaching program to determine what area needs to be taught again.

The tests are arranged in two four-page booklets, Diagnostic Number Tests 1 (involving addition, subtraction, simple multiplication and simple division), and Diagnostic Number Tests 2 (involving long multiplication and long division), and have a thorough teacher's manual for reference with them. The tests are arranged from an easy to more difficult format in an hierarchical pattern. On the scoring keys for the tests are complete descriptions of the test items.

The tests may be used with any age or grade where the test items meet the curriculum being taught. The tests should be administered without time limit.

After specific errors have been isolated, the teacher will need systematic materials to remediate the student's weaknesses.

The test manual indicates no data on the validity or reliability of the tests.

The specific test items are well-selected and the interpretive analysis of the type of test item is a valuable description.

SUMMARY

The reviews of these eighteen tests leave one with several impressions. First, there are many tests available but none are comprehensive enough to be used as a sole determiner of a difficulty in mathematics. At best, the tests represent general screening procedures that require detailed follow-up by interview.

Second, there is a lack of tests to be used at the secondary level. It is indeed unfortunate that more is not done to diagnose and correct difficulties students experience in algebra and geometry. No doubt many students with high mathematical potential could be salvaged if there were more diagnostic and prescriptive behavior at this level.

Third, from the variety of tests, it is easy to see the many factors involved in diagnostic-prescriptive procedures. Tests are only as useful if teachers utilize the information gathered in a positive manner and in conjunction with other techniques.

Finally, and of utmost importance, is the necessity to exercise caution with tests. There is a tendency among some educators to "overtest" students. Often this tendency is accompanied by a high degree of ability to "under-interpret" and "under prescribe." Diagnosis is the initial step. From this initial point, there follows a long and winding trail. Along the trail, there are crucial steps in terms of interviewing, prescribing, re-evaluating, re-teaching and then starting the process again.

We must remember that a diagnostic instrument is not a cure but rather an instrument for the curious. It is helpful in prescribing the medicine, but to be successful, the medicine must be made available to the patient and this must be a positive reaction between the patient and the medicine prescribed.

Chapter 7

RESEARCHING THE VALIDITY OF A DIAGNOSTIC TEST

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Validity for a diagnostic test should not be defined in the same way that validity for an achievement test usually has been defined, i.e., a correlation with a criterion measure. The validity for a diagnostic test should be defined as "user" validity. That is, if the test provides useful diagnostic information for remedial instruction, then the test is valid diagnostically.

The purpose of this study was to empirically examine the validity of a specific test, The Maryland Diagnostic Arithmetic Test (MDAT). This test was designed to help answer three questions about a child's learning in arithmetic: (1) What are the mathematical learnings which are present and which are absent? (2) At what degree of correctness or level of maturity in the child for concepts which are present? Are the concepts generalized over all domains? At what level is the child--symbolic, pictorial or three-dimensional? Does he translate across these levels? What processes does he use in arriving at results? (3) What are possible interdependencies among the concepts in arithmetic?

Since the MDAT is a paper and pencil instrument, it cannot be used alone to provide answers to all these questions. Only the symbolic and pictorial levels can be evaluated using the MDAT. There also is doubt about the validity of inferring process from the products of a paper/pencil instrument. These limitations of the MDAT were investigated using a different diagnostic procedure. This procedure involved some interesting stimulus tasks as referred to by Romberg (1979).

The questions that the MDAT can help answer are one and three. Strengths and weaknesses can be answered by a survey test, but how useful is it to know that a child is good at arithmetic applications but poor in concepts when it comes to planning for instruction for that child? The MDAT is not a survey test in its present form. It is made up of two forms, Primary and Intermediate, and each form has several sections with items relating to a cluster of content in arithmetic. For example, there are sections on computation, sections on models and meanings for operations and sections on sets and place value. The sections are categorized according to Wilson's Content Taxonomy (Wilson, 1978).

In attempting to assess a child's strengths and weaknesses in arithmetic, the information that is most useful is information that leads to remedial instruction. Thus, if you are able to write a prescription from the results of a diagnostic test and by following this prescription find that an

individualized remedial program is effective then the test has "user" validity. However, before such a study can be run, the test must be able to identify students who know a particular concept from those who do not know the concept.

The statistical procedure that was used to assess the diagnostic validity of the MDAT is based on a probabilistic model which provides a method for evaluating a domain-referenced test while taking into account "guessing" and "forgetting" by subjects.

The model is presented in detail in Dayton and Macready (1976) and its application in the assessment of mastery is presented in another source (Macready and Dayton, 1977).

The probabilistic model is based on dichotomously scored items (0,1) which are grouped into domains of items. The domain of items is constructed so that if a subject possesses the skill or concept being tested by a domain, he will then be able to correctly answer all questions in the domain. Conversely, if a student does not possess the skill or concept being tested by the items in the domain, he will then not be able to answer any of the items correctly in the domain. Thus, it can be seen that a domain should be made up of items which are testing a single skill or concept rather than a combination of skills or concepts.

Macready and Merwin (1973) have suggested that domains be constructed on the basis of homogeneity of item content and internal consistency of subject responses, so that the all-or-none assumption for domain mastery is more reasonable. The domains for the MDAT were determined a priori on the basis of the rational underlying Wilson's Content Taxonomy and Content Structures (Wilson, 1978).

The factors that are taken into account by the probabilistic models being used in this study are the errors made by students which account for total scores other than zero (for non-masters) or four (for masters on a four item domain). In the ideal situation, a domain is homogeneous and the population is made up of two disjoint sets of subjects, masters and non-masters. The subjects who are masters are able to do all four items in the domain correctly (producing a score of 4) while all the non-masters have a score of zero having incorrectly answered each item in the domain. In the case of 50% masters in the population, this situation would result in a symmetric bimodal distribution with no scores at 1, 2 or 3.

However, in most situations, this ideal result is not obtained. Some subjects who have acquired the skill somehow have a lapse of memory and rather than a score of 4, the score is 3 or 2. In like manner, some subject who has not acquired the concept is able to guess and rather than a score of 0, his score is 1 or 2. If one strictly adheres to a mastery score of four for the domain then, the subject who "forgot" will be misclassified as a non-master, which will be an error of classification. The model under discussion takes into account the probability that a master will "forget" a given item. This is referred to as beta, the omission or

"forgetting" error associated with each item. This error rate is a conditional probability. It is the likelihood that a subject who has acquired the concept will be unable to come up with the correct answer. It is an oversimplification to call beta the "forgetting" error since many factors are contributing to the lack of a correct response.

The other misclassification error that is possible for a mastery/non-mastery dichotomy is that of the non-master being classified as a master in the case where he has "guessed" well enough to produce a mastery score. This produces a "guessing" or intrusion error, referred to as alpha, an error rate associated with each item. Alpha is also a conditional probability. It is the likelihood that a subject who has not acquired the concept will be able to produce a correct answer on a specific item in the domain.

The alpha and beta error probabilities are estimated for each item in the domain under Model II, the model which is based on the assumption that these item error rates are different for each item in the domain. Under Model I, alpha error is assumed to be constant for all items in the domain, and beta is also assumed to be constant for all items. However, alpha and beta are not necessarily equal for items in the domain.

In addition to the estimated values for alpha and beta, each model estimates the proportion of masters in the population. This parameter is theta for both models.

In summary then, there are three parameters estimated under Model I: theta--the proportion of masters; alpha--the intrusion or "guessing" error; and beta--the omission or "forgetting" error. Model II estimates 2K + 1 parameters for a K item domain. These are: theta--the proportion of masters; $alpha_i$ for i = 1 to K - the intrusion error for each item; and $beta_i$ for i = 1 to K - the omission parameter for each item.

The fit of the model to the data is assessed by the standard (Pearson) chi-square goodness-of-fit test. The "expected" or predicted frequencies for each of the score patterns of zeros and ones for the domain are compared to the observed frequency for each pattern of zeros and ones. This Pearson chi-square statistic is evaluated as chi-square with degrees of freedom equal to 2^K-q-1, where q is the number of parameters estimated under the model (e.g., q = 3 in Model I). Values of p greater than .001 will be accepted as evidence of fit to the models where p is the right tail probability of the chi-square distribution.

Although several studies are now being carried out on the MDAT, this paper only examines a few domains of test items. Two areas of content that seem to be areas of difficulty for most children are place value concepts and meanings of the operations. Items testing these categories were given to fifth grade students in a public school in Maryland.

WRITE THE CORRECT NUMERAL IN EACH BLANK.

A) 6 ones and three tens = _____
B) 5 ones + 3 tens + 6 hundreds = _____
C) 3456 = _____ tens + _____ ones + _____ hundreds + _____ thousands
D) 4 ones + 7 hundreds = _____

Figure 1. Test Items for Domain 1

The results for Domain 1 with Model I are shown in Table 1.

Table 1

Model I Results for Domain 1

Paremeter	Estimated Value
Theta	.69
Alpha	.03
Beta	.13

Chi-square Summary

SCORE	OBS N	PRED N	CHI-SQUARE
0	23	22.91	.00
1	3	3.21	.01
2	7	4.11	2.02
3	13	18.51	1.64
4	35	32.36	.23

CHI-SQUARE = 3.91 WITH DF = 1

Note that for domain 1, theta = .69 or for this population of fifth graders, 69% are masters of the domain. The absolute fit of this data to the model is not tenable, P-value < .05. The comparison of observed N (OBS N) values for each score frequency with the predicted N (PRED N) values shows some discrepancies for scores of 3 and 4. Note that this lack of fit contributes heavily to the chi-square. Also, note the low (.03) value for the intrusion parameter, a not-unexpected finding given the item format. One would expect little "guessing" to occur on these items.

Table 2

Model II Results for Domain 1

Parameter	Estimated Value
Theta	.68
Alpha 1	.07
Alpha 2	.00
Alpha 3	.04
Alpha 4	.00
Beta 1	.06
Beta 2	.06
Beta 3	.28
Beta 4	.11

Chi-square Summary

PATTERN	OBS N	PRED N	CHI-SQUARE
0000	23	22.96	.00
1000	2	1.92	.00
0100	0	.09	.09
1100	3	1.52	1.43
0010	1	.92	.01
1010	0	.31	.31
0110	0	.24	.24
1110	3	4.01	.25
0001	2	.04	.04
1001	0	.73	2.24
0101	1	.72	.11
1101	9	12.02	.76
0011	1	.11	6.82
1011	0	1.91	1.91
0111	1	1.90	.43
1111	35	31.59	.37

CHI-SQUARE = 15.01 WITH DF = 6

The results under Model II are in agreement with the results of the first model, about 68% are masters of the domain. What is of interest in the Model II results is the large omission value (Beta 3) for item three. In looking back at what item three requires of the student, it is not readily apparent why so many students would miss the item. However, in examining the actual test papers, the test item had been typed so that the item was split between two lines and the last blank was on the second line. Thus, many students did not fill in this blank and missed the item. Therefore, the relatively large beta 3 value is due to poor item format, not to a lack of item homogeneity. In looking at item patterns under the chi-square summary, it can be seen that there are nine students with a score pattern of 1101, students who probably are masters of the domain. The absolute fit of

this model is also not tenable, p-value < .05. The domain probably could be improved with a change in item 3.

The results for the remaining four domains are organized by content objectives (see Wilson, 1976) which define the domain parameters. Student's knowledge of the following content objective was assessed by the test items in domain 2.

Content Objective for Domain 2:

> Any whole number expressed in a non-standard form may be named by a standard numeral by renaming ten of the ones /(or tens, or hundreds, etc.) or a multiple of ones/ as one (or more) ten /(or hundreds, or thousands, etc.) or a multiple of ten/ and adding this to the number of tens (or hundreds, or thousands, etc.) expressed in the non-standard form.
>
> E.g., Non-Standard Forms
>
> 2 hundreds + 14 tens + 16 ones + 3 hundreds + (10 tens + 4 tens) + (10 ones + 6 ones) = (3 hundreds + 1 hundred) + (4 tens + 1 ten) + 6 ones = 456 (Standard Name)

The items in domain 2 are in Figure 2.

WRITE THE CORRECT NUMERAL IN EACH BLANK.

A) 7 tens + 15 ones = _____
B) 42 tens and 23 ones = _____
C) 965 = 15 ones + _____ tens + 9 hundreds
D) 2163 = _____ hundreds + 16 tens + 3 ones

Figure 2. Test Items for Domain 2

The Model I data for domain 2 are in Table 3.

Table 3

Model I Results for Domain 2

Parameter	Estimated Value
Theta	.46
Alpha	.09
Beta	.40

Chi-square Summary

SCORE	OBS N	PRED N	CHI-SQUARE
0	32	31.95	.00
1	18	18.15	.00
2	15	14.53	.02
3	12	12.69	.04
4	5	4.68	.02

CHI-SQUARE = .08 WITH DF = 1

Note the extremely good fit of the data to Model I (P-value .05) under the assumption of constant intrusion and omission parameters across the items of the domain. This domain estimates about 45% masters in the population under Model I and 40% masters under Model II (See Table 4).

Table 4

Model II Results for Domain 2

Parameter	Estimated Value
Theta	.40
Alpha 1	.31
Alpha 2	.00
Alpha 3	.07
Alpha 4	.00
Beta 1	.00
Beta 2	.24
Beta 3	.44
Beta 4	.73

CHI-SQUARE = 2.46 WITH DF = 6

Sentences used in the other four items of the domain are shown in Figure 4.

DRAW A RING AROUND THE BEST WAY TO SOLVE THIS PROBLEM, $\square \div 16 = 12$.

A) $12\overline{)16}$ B) $16\overline{)\square}$ C) $\begin{array}{r} 16 \\ \times\ 12 \\ \hline \square \end{array}$ D) $\begin{array}{r} 16 \\ -\ 12 \\ \hline \square \end{array}$

Figure 3. Test Item 2 for Domain 3

1. $7 \times 5 = \square$
2. $\square\overline{)48}$
3. $18 \times \square = 90$
4. $85 \div \square = 17$

Figure 4. Test Items 1, 3, 4 and 5 for Domain 3

The model II data analysis appears in Table 5.

Table 5

Model II Results for Domain 3

Parameter	Estimated Value
Theta	.24
Alpha 1	.50
Alpha 2	.42
Alpha 3	.17
Alpha 4	.08
Beta 1	.00
Beta 2	.34
Beta 3	.99
Beta 4	.42
Beta 5	.16

CHI-SQUARE = 8.78

WITH DF = 20

The item appearing as item 2 in this domain was of particular interest because of the omission parameter. The estimated parameter value has exceeded the 0-1 limits for the parameters. Note that the placeholder appears as the first term in the open sentence which makes this open sentence unlike the other sentences in the domain. It was hypothesized that the position of the placeholder or variable in the item was a factor to be considered in defining the domain (McMaster, 1976; Grouws & Good, 1976) so the following items were constructed and tested as a domain with the same group of fifth graders. As with the items in domain 3, the same stem was used with the four open sentences listed above the stem. The test items for domain 4 appear in Figure 5, and the results appear in Table 6.

$\square \times 9 = 63$ $\square \times 16 = 64$ $\square \div 4 = 9$ $\square \div 4 = 24$

To find the missing number, I would _____.

Figure 5. Test Items for Domain 4

Table 6

Model I Results for Domain 4

Parameter	Estimated Value
Theta	.59
Alpha	.05
Beta	.23

Chi-Square Summary

SCORE	OBS N	PRED N	CHI-SQUARE
0	26	26.04	.00
1	7	6.89	.00
2	8	9.01	.11
3	21	19.14	.18
4	15	15.91	.05

CHI-SQUARE = .3475 WITH DF = 1

The fit of the model for this domain of items is excellent, although the number of scores at three is unusually large. Further investigation is needed here.

The last domain (Domain 5) is of interest because while the number of masters is extremely low (.03), (See Figure 6 and Tables 7 and 8) and the domain does not fit the model, the 17 scores of two clearly indicate a large

group of students who have partial knowledge, i.e., they are able to identify factors and products in a multiplication sentence, but not in a division sentence.

DRAW A RING AROUND ANY NUMERAL THAT IS A FACTOR AND PUT AN X ON ANY PRODUCT IN THESE PROBLEMS.

A) $3 + 4 = 7$ C) $17 - 8 = 9$

B) $6 \times 2 = 12$ D) $24 \div 3 = 8$

Figure 6. Test Items for Domain 5

Table 7

Model I Results for Domain 5

Parameter	Estimated Value
Theta	.07
Alpha	.00
Beta	.75

Chi-Square Summary

SCORE	OBS N	PRED N	CHI-SQUARE
0	54	27.67	25.06
1	4	30.33	22.86
2	17	15.33	.19
3	0	3.42	3.42
4	2	.29	10.19

CHI-SQUARE = 61.73 WITH DF = 1

Table 8

Model II Results for Domain 5

Parameter	Estimated Value
Theta	.03*
Alpha 1	.28
Alpha 2	.22

Chi-Square Summary

PATTERN	OBS N	PRED N*	CHI-SQUARE
0000	54	41.76	3.59
1000	4	16.24	9.23
0100	0	12.24	12.24
1100	17	4.76	31.47
1111	2	2.00	.00

CHI-SQUARE = 56.53 WITH DF = 6

*Only non-zero parameter estimates and predicted frequencies are shown

---

Thus, it is left up to the test constructor whether the items should be reconstructed into two domains or left unchanged with a note about the probable interpretation for a score of two. Item 1 in domain 5 is the identification of the product 12, item 2 is the identification of either factor 6 or 2, item 3 is the identification of the product 24 and item 4 is the identification of either factor 8 or 3.

The items reported on here are only a small sample of the items being tested in this research project. The item domains will be tested on a larger group of students following the restructuring of the item domains. Homogeneously grouped items make it possible to test over several different domains for master. Thus, relationships between concepts and clusters of concepts will be tested. For example, test results could be used to investigate questions such as: 1) Is there a relationship between what a student knows about sentence solving, verbal problem solving and models for the operation? or, 2) Is there a relationship between students who are masters of computation and students who have mastered place value concepts? These questions are under consideration as part of the larger research project. A more complete report on these results can be found elsewhere (Sadowski, 1978).

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Chapter 8

A DIAGNOSTIC MODEL FOR USE BY CLASSROOM TEACHERS

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INTRODUCTION

A great deal of attention has been given to the development of clinical models for diagnosing students' learning difficulties in mathematics. Although clinical models have the potential for generating significant research data related to the nature of the learning difficulties in mathematics, such models do not provide practical assistance to the classroom teacher for making diagnoses on a day-to-day basis. The implementation of a clinical model is expensive both in terms of required personnel resources and student time. And further, the diagnostician implementing the model must have extensive training in techniques of collecting and analyzing data. The restrictions on the utilization of clinical models do not minimize the importance of their role in the area of diagnosis of mathematical learning difficulties, but rather emphasize the need for diagnostic models which are practical for use in the classroom.

A classroom diagnostic model should have one essential characteristic of the clinical model: the model must take into consideration the nature of the student's performance on a diagnostic test. That is, the model should be sensitive to the underlying causes of a student's errors, thereby, enabling the teacher to prescribe remedial activities which are explicitly related to the nature of the student's learning difficulty. In addition, a classroom model should satisfy at least the following practical criteria: (1) the student-testing time should be kept to a minimum; (2) the model should be usable by a teacher who has only limited training in diagnostic/prescriptive techniques; (3) the model should encompass at least 80% of the diagnostic problems which a teacher encounters on a day-to-day basis. If these three criteria can be incorporated into a single model which also indicates when unusual or difficult cases should be handled on an individual basis by the classroom teacher or a mathematics specialist, the implementation of diagnostic/prescriptive teaching strategies in the classroom will be greatly enhanced.

The purpose of this paper is to suggest a model for designing a classroom diagnostic instrument which satisfies the above criteria. The format of the model is similar to that used in some home medical guides which provide a list of symptoms in one column, identification of the illness in the second column and recommended medication in the third column. The format of the proposed diagnostic instrument is shown on the next page in Figure 1.

Analysis Student's Work	Probable Cause	Recommended Remediation

Figure 1. Format of Diagnostic Model

An application of the basic diagnostic model to a specific situation is provided at the end of this paper. The purposes for presenting this prototype instrument are to illustrate procedures for developing an instrument according to the format of the basic model, to explain how the instrument might be used by classroom teachers, and to identify areas of needed research.

Instruments constructed according to the format shown in Figure 1 above should not be designed to cover every conceivable error which might be made by students at every grade level. Rather each instrument should address only the errors which are frequently made by students at a specific grade level who are studying a particular curriculum. Thus, it might be necessary to modify the specific contents of a given instrument for use with students at other grade levels and/or in schools with different curricula.

DEVELOPMENT OF A PROTOTYPE INSTRUMENT

The prototype instrument presented in this paper was designed to assess the difficulties students in the upper elementary grades have with the addition and subtraction algorithms. The development of the instrument began with a review of research studies on error analyses to determine the types of errors most frequently made by students. Next, a short twelve item test covering addition and subtraction of one and two digit numbers with and without regrouping was constructed. This test was administered to 250 students in the upper elementary grades. These students were enrolled in three schools. Each school had a different mathematics curriculum. There were major differences among the student body compositions of the three schools. An analysis of the students' performances revealed that approximately 95% of the errors could be classified in one of six general categories. Each category was tentatively assigned a descriptive name which reflected a probable explanation for the error pattern. The six error categories were: 1) memory; 2) facts; 3) perseveration; 4) reading; 5) zero; 6) multiplication effect.

The next phase of the development process was to determine whether or not there was a predominant cause for each error pattern. The first step in this validation process was to administer the twelve item test to another sample of students in the upper elementary grades. This sample included all of the 133 fourth and 162 fifth grade students in two elementary schools. Even though the schools used different textbook series and there were significant differences in the student population of the two schools, no major differences were detected in the nature of the students' responses from the two schools.

The data obtained from this administration of the test are reported in Table 1.

Table 1

Distribution of Errors by Categories

Error Category	Number of Students		
	Fourth Grade (133)	Fifth Grade (162)	Total
Memory	5	8	13
Facts	16	16	32
Perseveration	2	4	6
Reading	16	20	36
Zero	12	5	17
Multiplication Effect	14	18	32
Random	6	8	14

An analysis of the data indicates that approximately 50% of the students in the combined population were in possible need of remedial instruction in addition and subtraction of whole numbers.

Of these students, approximately 90% could be tentatively identified with one of the six error categories. (The perseveration category was retained because it had been noted earlier that a large number of errors could be attributed to this cause, particularly among students at the sixth and seventh grade level who took the test.)

The next step of the validation process was to select from among the students in each error category a small sample, three to eight in each school, for follow-up interviews. The purposes of these interviews were to establish the consistency of the students' errors and to ascertain a probable cause for the incorrect responses. A specific interview was developed for each of the six categories. Each interview included problems similar to those missed on the diagnostic test and provided tasks which assessed the learning difficulty under investigation in a different context. A structured set of questions were asked in each interview; however, the interviewer could ask probing questions as the situation warranted. The data obtained from the interviews indicated that for 85% of the students interviewed, the errors made on the diagnostic test could possibly be attributed to the cause tentatively identified in the classification of error patterns. This data seemed to be sufficient justification for developing a collection of six remedial programs, each explicitly related to probable causes for an error pattern.

Appropriate remedial activities were then prescribed for the interviewed students in each category. The remedial instruction was terminated when a student's performance on a practice test demonstrated that the

learning difficulty had been corrected. The length of time students were engaged in the remedial activities ranged from 2-5 hours. A delayed post-test (four weeks) was given to assess the long range effectiveness of the remedial instruction. Data which describe the students' performance on the posttest are reported in Table 2.

Table 2

Effectiveness of Remedial Instruction: Posttest Results

Error Category	Instruction was Effective	
	Yes	No
Memory	6	1
Facts	8	1
Perseveration	3	0
Reading	8	2
Zero	6	2
Multiplication Effect	8	0

EVALUATION OF PROTOTYPE INSTRUMENT

The results as reported in Table 2 indicate that the remedial instruction was effective for most of the students. There are a variety of reasons why the remediation may not have been effective for all students. There were perhaps inadequacies in the nature of the remedial activities. It is possible that the remedial instruction did not provide sufficient time to change behaviors which had been practiced over a period of several years. Affective factors such as motivation and behavior problems could explain why some students continued to make the same mistakes on the posttest.

One should not infer from the data presented in Table 2 that the remedial activities had corrected all of the student's problems, because it was noted on the posttest that although one error pattern had been corrected a second error pattern emerged in a few cases. In particular it was noted that several students who had not shown signs of a perseveration problem on the first test did exhibit this difficulty on the posttest.

FURTHER DEVELOPMENT OF THE DIAGNOSTIC INSTRUMENT

The prototype instrument was developed to investigate the feasibility of constructing a classroom diagnostic/prescriptive instrument which reflects the concept of the basic diagnostic model. The results of this preliminary study suggest that this concept of the classroom diagnostic model provides a realistic plan for constructing diagnostic/prescriptive instruments for classroom use. The prototype instrument, however, should not be viewed as

a finished product. Investigations associated with the development of the instrument have raised several critical questions for further study. Among these research issues are the following:

1. The remedial suggestions provided in the prototype instrument are not precisely stated in terms of the amount of instruction which should be provided. The effectiveness of the model could be enhanced if it could be stated with 95% certainty that a prescription of a specific number of certain types of exercises for so many days would correct the causes associated with the given class of errors. Such data is not now available, but the existence of information of this type would make this concept of diagnostic teaching more acceptable to classroom teachers.

2. Six different error categories were identified in the prototype instrument. Consequently, there are six suggestions for remedial activities. The necessity for six different remedial programs was not verified. This aspect of the model should be investigated further. If in fact one remedial prescription is effective in correcting the causes of two or more error patterns, then the format of the instrument itself and the ultimate utilization of the instrument would be greatly simplified.

3. Two or more different types of remedial activities are suggested for each error category. The extent to which each type of activity contributed to the correction of the underlying causes associated with an error pattern was not determined. This question must be resolved in order to maximize the effectiveness of the remedial instruction.

4. The prescribed remediation was designed to correct errors associated with the addition and subtraction algorithms. Assuming that this instruction is effective for the stated purpose, a broader question should be considered: would this instruction remediate and/or prevent similar errors associated with other computational skills?

5. The basic concept of the classroom diagnostic model has been tested only with addition and subtraction of whole numbers. A question remains as to whether or not similar instruments could be constructed for other skills such as multiplication and division of whole numbers; or addition and subtraction of fractions. If such instruments can be developed, then the concept of the model would provide a universal format for the construction of diagnostic/prescriptive instruments.

6. The diagnostic test utilized in the prototype instrument consisted of only 12 relative simple items. The test in its present form has one particular advantage: it is easy to administer. But this feature should not restrict a search for a test which would provide better diagnostic information.

7. The prototype instrument has been field tested only by those persons who participated in its development. The ultimate question which must be addressed is whether or not classroom teachers with limited training in diagnosing mathematical learning difficulties can and will use an instrument of this type. This aspect of the validation of the instrument, however, should not be undertaken until the issues raised above have been investigated further.

The above issues are raised for two primary purposes. First, to provide some indication of the magnitude of the tasks which face researchers in the area of diagnostic/prescriptive mathematics. And secondly, to identify research questions for mathematics educators to consider as they pursue their research in the area of diagnostic and prescriptive mathematics.

HOW TO USE THE INSTRUMENT

The procedures for using the instrument are as follows. First, the diagnostic test is given to the students and graded. The second step is for the teacher to analyze a student's errors by identifying the error pattern, if possible- with one of the six error descriptions provided in the model. Analyses of students' performances are contained in the upper left hand corner of each page. If only one identifiable error pattern is noted, then the teacher determines a probable cause for the learning difficulty (upper right hand corner) and then prescribes a remedial program based on the recommended activities (bottom part of the page). If a student's work exhibits two or more error patterns, then the teacher designs an appropriate remedial program which blends activities related to each of the learning difficulties.

The prototype instrument is provided on the next six pages of this article, one page for each error category.

ANALYSIS OF STUDENT'S RESPONSE	CAUSE
1. Addition and subtraction problems without regrouping are correct. 2. Addition and/or subtraction problems with regrouping are incorrect. The one's digit is correct, but the ten's digit is one less than the correct answer. Example: 47 + 23 = 60 ; 68 + 4 = 62 ; 72 - 6 = 76 ; 50 - 17 = 43	Memory After correctly adding or subtracting the numbers in the one's column, the student forgets to change the ten's column to show the regrouping step.

REMEDIATION

1. Provide short term memory excercises. For example, dictate a sequence of 2, 3, or 4 numbers and have the students record the numbers after the sequence is completed. Repeat several times. As a variation, have the students record the numbers in reverse order.

2. Explain carry helper (addition) or cross-out helper (subtraction) in order to help students remember the regrouping step.

Example:

$$\begin{array}{r} (1) \\ 26 \\ +38 \\ \hline 64 \end{array} \qquad \begin{array}{r} 2\ 12 \\ \not{3}\not{2} \\ -19 \\ \hline 13 \end{array}$$

Prepare practice worksheets. Remind students to use the helpers.

3. Continue drill activities but do not insist that the students use the helpers.

ANALYSIS OF STUDENT'S RESPONSE	CAUSE
1. The student adds or subtracts as indicated. 2. He (she) has no errors in regrouping. 3. The one's and/or ten's digit is one or two more (less) than the correct answer. Example: $\begin{array}{r} 76 \\ +\ 8 \\ \hline 83 \end{array}$ $\begin{array}{r} 34 \\ +19 \\ \hline 52 \end{array}$ $\begin{array}{r} 56 \\ -\ 8 \\ \hline 49 \end{array}$ $\begin{array}{r} 72 \\ -16 \\ \hline 47 \end{array}$	Facts The student's difficulty is determining the basic facts. The errors may be attributable to incorrect computing schemes (fingers or tally marks) or to incorrect memorization.

REMEDIATION

1. Begin by discussing with the students why memorization of the facts is important.

2. Teach an efficient process for computing facts. To teach the hard addition facts, the following technique could be taught.
   (a) Explain how 9 can be shown on one hand as 4-fingers; 8 as 3-fingers; 7 as 2-fingers; 6 as 1-finger and 5 as 0-fingers. Practice this skill.
   (b) Explain how 7 + 9 can be shown as 2-fingers on one hand and 4-fingers on the other hand, and that the sum when added to ten is the sum of 7 and 9. Continue explanation with other examples. Practice using this technique.
   (c) Extend practice to include 2 and 3 digit addends. Remind students to use the new techniques.

   A similar procedure can be used to compute the hard subtraction facts.
   (a) Explain that in subtraction, 9 can be shown on one hand as 1-finger; 8 as 2-fingers; 7 as 3-fingers; 6 as 4-fingers and 5 as 5-fingers. Practice this skill.
   (b) Explain how 14-9 can be computed by showing 4 fingers on one hand (one digit) and 1 finger on the other hand (the 9) and adding the fingers to find the difference. Continue with other examples, then provide practice.
   (c) Extend practice to include subtraction with 2 and 3 digit numbers. Remind the students to use the technique.

3. Provide practice exercises which include both addition and subtraction problems.

4. Give short timed tests, 10-20 problems. Chart the student's progress-accuracy and time.

ANALYSIS OF STUDENT'S RESPONSE

1. Addition problems are correct with and without regrouping.

2. The last (4 or 5) subtraction problems may be correct.

3. The student added the first few (or all) of the subtraction problems.

Example:

47	57	68	56
+23	- 4	-24	- 8
70	61	92	48

CAUSE

Perseveration — The child has difficulty in shifting from one stimulus (addition) or another (subtraction).

REMEDIATION

1. Discuss with students the importance of considering each problem carefully (stop, look, think and then solve).

2. Make a list of symbols: + - + - - - + - - + - + + + + - + - + - +. Ask the students to read the symbols. Continue practice by encouraging the student to read as rapidly as possible.

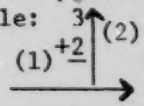
3. Extend practice on reading symbols by having the student only read the operation signs in a problem exercise like:

6	7	6	3	5	4	3
+4	+5	-5	+2	-1	-1	+4

4. Dictate a sequence of operations (plus, plus, minus, ..., etc.). Have the students record the sequence.

5. Emphasize the aspect of reading the operation symbol before solving the problem. Give 6 problems to the students (add., subt.) and tell them to stop at each problem, look carefully, think, read the operation and then solve the problem.

6. Have the students practice reading problems in this order: first the operation, then numbers from bottom to top. Example:



7. Provide drill exercises which contain a mixture of addition and subtraction problems. Remind students to stop, look, think and then solve.

ANALYSIS OF STUDENT'S RESPONSE	CAUSE
1. Addition problems are correct with and without regrouping. 2. Subtraction problems without regrouping are correct. 3. In solving subtraction problems with regrouping, the student avoids the regrouping step by subtracting the smaller number from the larger number. Example: 47 +23 = 70; 68 −24 = 44; 56 − 8 = 52; 72 −16 = 64; 56 −49 = 13	Reading The student does not identify subtraction problems which "cannot" be done. The student may not be aware of the existence of such problems and/or the student incorrectly reads the problem as a problem which can be solved and/or the student does not distinguish between two verbalizations such as '9 take away 2' and '2 take away 9.'

REMEDIATION

1. Give the students a set of subtraction problems like: 3 −1; 2 −3; 5 −7; 3 −1

   Have the students identify the problems which cannot be done. Discuss why some subtraction problems cannot be solved. Provide practice exercises, where the students identify (circle) the "cannot" do problems.

2. Explain how to read subtraction problems. Point out that a problem may be read in several ways, some are correct and some incorrect. Have the students practice reading subtraction problems. Encourage the students to be consistent in the way they read problems. Include in this activity "cannot" do problems and have them say "can't do" after reading such problems.

3. Dictate problems using a variety of statements and have the students record the problems. "Cannot" do problems should be identified.

4. Explain that some subtraction problems involve a "cannot" do step. Give the students a set of 2 and 3 digit subtraction problems, some without regrouping and have them circle the "can't" do steps, then solve the problems.

5. Provide practice with and without regrouping. Remind students to look for "can't" do steps.

ANALYSIS OF STUDENT'S RESPONSE

1. Addition problems that do not involve zero are correct with and without regrouping.

2. Subtraction problems that do not involve zero are correct with and without regrouping.

3. In solving subtraction/addition problems, the student answers "0" or "n" for the zero step.

   Example:

$$\begin{array}{rrr} 47 & 72 & 50 \\ +23 & -16 & -17 \\ \hline 70 & 54 & 47 \end{array}$$

/40

CAUSE

Zero The student's attention is focused on one generalization about zero...n x 0 = 0, n + 0 = 0...This one generalization is applied to all problems involving a zero.

REMEDIATION

1. Give the students a set of problems like:

$$\begin{array}{rrrrrrrr} 0 & 0 & 3 & 0 & 7 & 4 & 0 & 0 \\ -3 & +0 & -0 & -0 & -0 & +0 & -7 & +4 \end{array}$$

   Have the students identify the problems which cannot be done. Discuss why some subtraction problems involving zero cannot be solved. For example, "You can't take 3 from zero." Provide practice exercises where the students identify (circle) the "cannot" do problems involving zero.

2. Provide a worksheet with problems like:

$$\begin{array}{rrrrr} 50 & 60 & 37 & 50 & 67 \\ -23 & +25 & +40 & +20 & -20 \end{array}$$

   and have a student read each problem and explain why it can or can't be worked. In the discussion, point out 2 types of subtraction problems: (a) Those with a "can't" do step $\begin{array}{r} 50 \\ -23 \end{array}$

   (b) Those without a "can't" do step $\begin{array}{r} 67 \\ -20 \end{array}$

3. Give the students a set of 2 and 3 digit subtraction (addition) problems, have them circle the zero step and then solve the problems. Discuss the results.

4. Provide practice with and without regrouping in subtraction (addition) involving zero. Remind students to look for "zero step."

ANALYSIS OF STUDENT'S RESPONSE	CAUSE
1. Addition and subtraction problems with two 2-digit numbers are correct. 2. In addition (subtraction) with a 2-digit and 1-digit numbers, the student (adds or subtracts) the 1-digit number to (from) both the ones and tens columns. Example: $\begin{array}{r} 72 \\ +\ 4 \\ \hline 116 \end{array}$ $\begin{array}{r} 31 \\ +24 \\ \hline 55 \end{array}$ $\begin{array}{r} {}^{1}76 \\ +\ 8 \\ \hline 164 \end{array}$ $\begin{array}{r} 57 \\ -\ 4 \\ \hline 13 \end{array}$ $\begin{array}{r} {}^{5}\not{6}{}^{12}2 \\ -\ 4 \\ \hline 18 \end{array}$	Multiplicative Effect The student is responding to the format rather than to the meaning of the problem. This error may be a result of the student's experiences with multiplication or a belief that it is necessary for two numbers to be added (subtracted) in each column.

REMEDIATION

1. Write the problem $\begin{array}{r} 54 \\ +\ 9 \\ \hline \end{array}$ on the board and ask students to read the problem and then explain step-by-step how to work the problem. Repeat using similar problems like:

$\begin{array}{r} 54 \\ \times\ 9 \\ \hline \end{array}$ $\begin{array}{r} 57 \\ \times\ 4 \\ \hline \end{array}$ $\begin{array}{r} 57 \\ \times\ 4 \\ \hline \end{array}$ $\begin{array}{r} 35 \\ \times\ 4 \\ \hline \end{array}$ $\begin{array}{r} 36 \\ +\ 4 \\ \hline \end{array}$

2. Provide a worksheet with problems like: $\begin{array}{r} 28 \\ +\ 2 \\ \hline 50 \end{array}$ $\begin{array}{r} 74 \\ +\ 5 \\ \hline 129 \end{array}$ $\begin{array}{r} 67 \\ -\ 4 \\ \hline 23 \end{array}$

   Ask students to circle those which have been worked incorrectly, have students explain the errors and then correct the problems.

3. Provide worksheets to reinforce addition, subtraction, and multiplication of 2-digit and 1-digit numbers.

Chapter 9

USING PIAGETIAN TASKS IN CLASSROOM DIAGNOSIS

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The modern elementary teacher is trained to develop mathematics programs that allow each student to maximize his/her learning. Diagnosis, the determination of each child's learning strengths and weaknesses, is crucial and central to this endeavor. As diagnostic teachers of mathematics, each of us is constantly seeking answers to a variety of questions, among them are:

1) What mathematical learnings are present and absent in this child?
2) What are the possible relationships and interdependencies among those learnings which are present and those which are absent?
3) How correct are those learnings which are present and are the learnings mature enough to generalize to other mathematics?

The teacher has many options for determining answers to these questions. The most frequently used procedure is to administer a standardized, a non-standardized, or a teacher-made paper and pencil test. Such tests usually provide important information concerning questions 1 and 2 above. They do not provide adequate information concerning the level of maturity a child has with regards to a particular mathematical concept. Neither do they give much insight into how the child is thinking or relating bits of information. This type of information must be obtained through different means.

During the last ten years, much has been said about the value of supplementing paper and pencil tests with other diagnostic procedures. The "interview," as a diagnostic procedure, has proven to be an effective way of revealing the type of reasoning the child is using, thereby providing insight into fallacies in logic, incorrect and immature mathematical concepts, and incomplete understanding of algorithms and computational procedures. Many types of procedures are named by the term "interview." One typical interview format is that found in the classical Piagetian tasks. In these tasks, a problem situation is presented to the child and through skillful probing by the examiner, is led to respond and to give reasons for his replies. From this type of dialogue, the examiner makes inferences about the child's knowledge of the problem. "Piagetian-type" tasks provide an excellent format for a teacher to use in determining what meanings a child really has of specific mathematics concepts and procedures. The information gathered through such an "interview," when used in conjunction with information obtained through more traditional means, allow the teacher to develop a program based on the diagnosed strengths and weaknesses of the child.

A set of eighteen "Piagetian-type" tasks are given below along with directions for administration and materials needed. It is also indicated on each task what mathematical concept or principle is being tested. These tasks cover classification using both one and two attributes, linear seriation, number conservation, cardinal-ordinal number integration, one-to-one correspondence, meaning of commutativity, and meaning of a fraction. With a little effort, "arithmetic" tasks requiring the same underlying thought patterns can be developed.

The preceding tasks are but a small sample of possible tasks available for diagnostic purposes. Additional tasks may be found in Copeland (1974), Piaget (1952), Inhelder and Piaget (1964) and numerous publications (NCTM, 1971; NCTM, 1975). Also, Tasks 10, 11, 12, 13 were developed as part of a research project by Greenbaum (1975).

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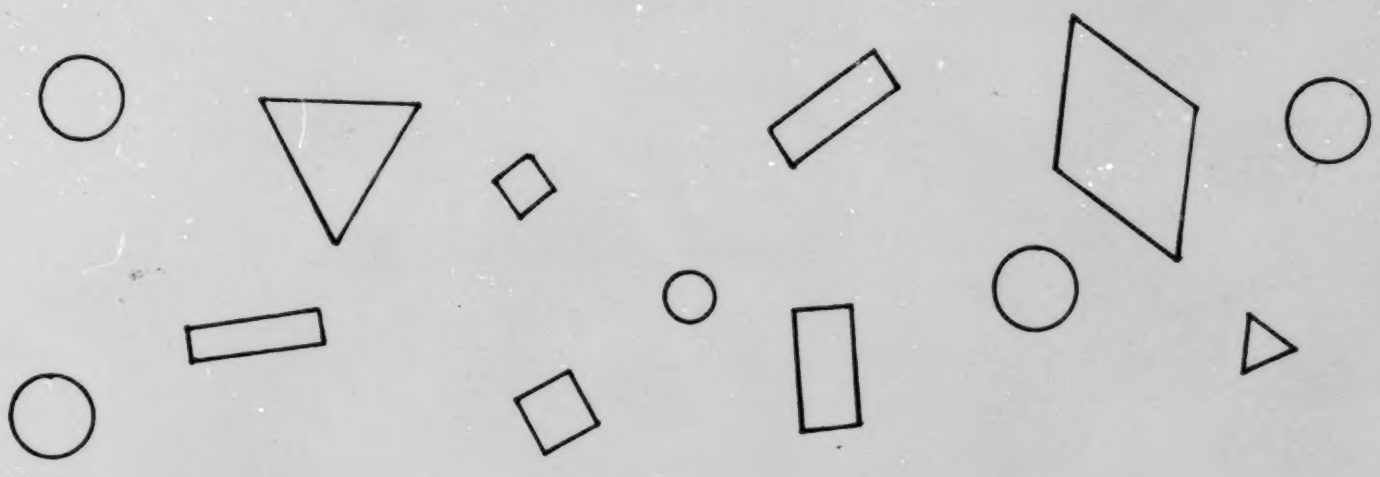
TASK 1

Classification using one attribute (Child determines and uses his rule for grouping)

Materials: Collection of geometric pieces having attributes of color, shape, size, etc.

Directions: Place the geometric pieces before the child as in the diagram below.



Ask, "Can you place all blocks together that should be together?" After the child's performance, say, "Why did you do it this way?" After an explanation, ask "Can you do it another way?"

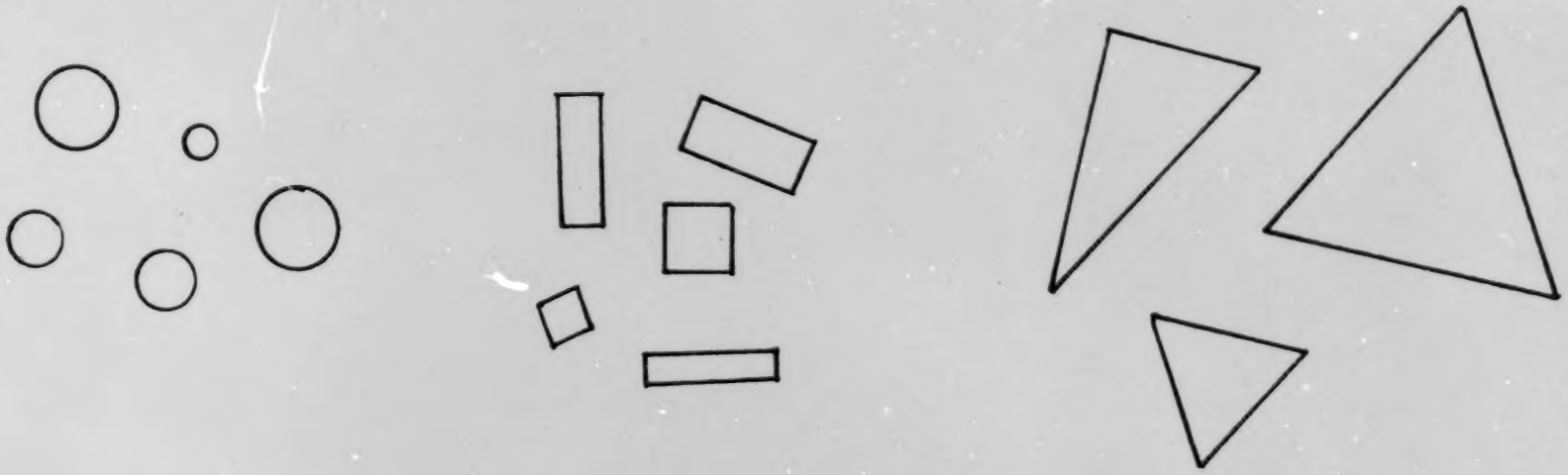
Variations: Use sticks and group on basis of length.
Use set cards having many sets of different cardinalities to group on numerousness.

TASK 2

Classification using one attribute (Child discovers rules used in grouping)

Materials: Collection of geometric pieces having attributes of color, size, shape, etc.

Directions: Teacher forms groups on basis of an attribute (as in the diagram below) and then says,



"I have made some groups for you. Can you tell me why I have all of these blocks here in this pile (pointing to a pile), in this pile and in this pile?"

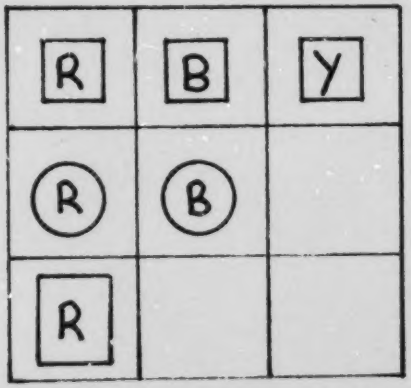
Variations: Group blocks on the basis of color
Group blocks on the basis of size
Group sticks on the basis of length
Group sets on the basis of numerosity

TASK 3

Classification using two attributes (Intersection of Sets)

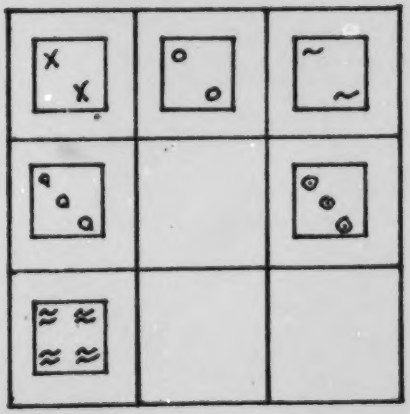
Materials: Set of attribute blocks
Piece of tagboard marked off into a 3 x 3 matrix

Directions: Teacher begins by placing blocks in the matrix as shown below:



Have the child fill out the matrix. Then ask, "Why did you put those pieces in the empty spaces?" Give the child a chance to explain his choices.

Variations: Set cards may be used instead of blocks (as below).
A variety of relational categories may be intersected using this format.

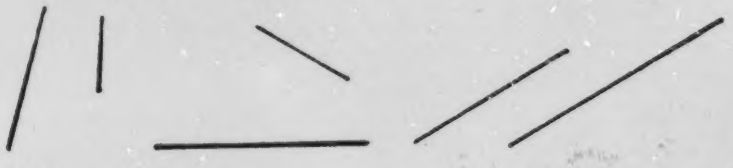


TASK 4

Seriation of linear objects

Materials: Set of six dowel rods (sticks) differing in length by 1/8"
Set of six strings differing in length by 1/8"

Directions: Place the six sticks on a table before the child.



Ask, "Find the longest stick," "Find the shortest stick." When this has been done, say, "Put these sticks in order from shortest to longest (longest to shortest)."

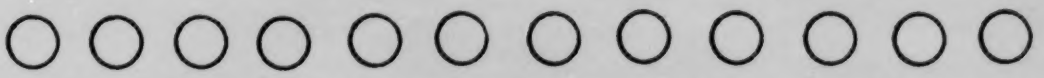
Variations: Using set cards, arrange the cards in order from set having least to set having most.
Using blocks to order on size
Using cans to order on capacity

TASK 5

Conservation of numerousness

Materials: 12 red counters
12 blue counters

Directions: Give the child one set of counters. Ask the child to place one counter down for every counter you place down, as follows.

Teacher 

Child

Now ask, "Do we have the same amount of counters?" "Why do you say so?"

Transform one set as follows:

Teacher

Child

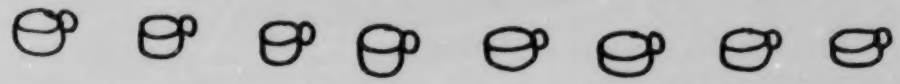
Now ask, "Do you have the same amount of counters as I?"
"Do you have more counters than I?"
"Do you have less (fewer) counters than I?"
"Why do you say so?"

TASK 6

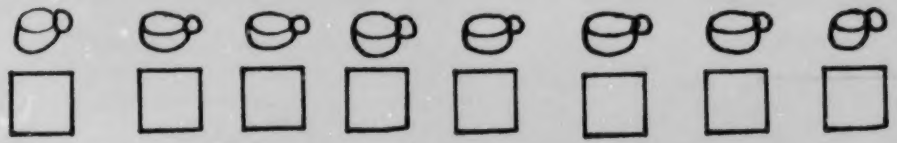
One-to-one correspondence

Materials: 8 paper cups
10 napkins
15 plastic spoons

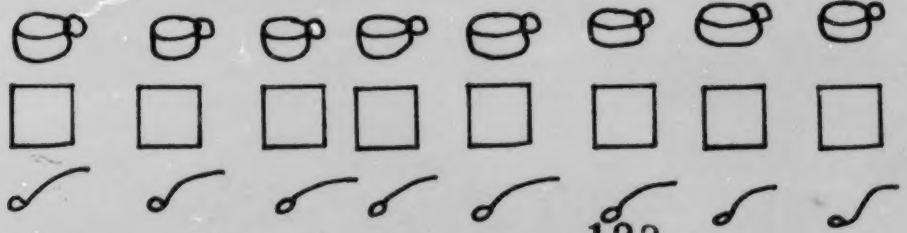
Directions: Pretend that we are going to have a party. Seven friends have been invited, so along with yourself there will be eight people. Give each person a cup.



Now give each person a napkin to go with the cup.



Now give each person a spoon



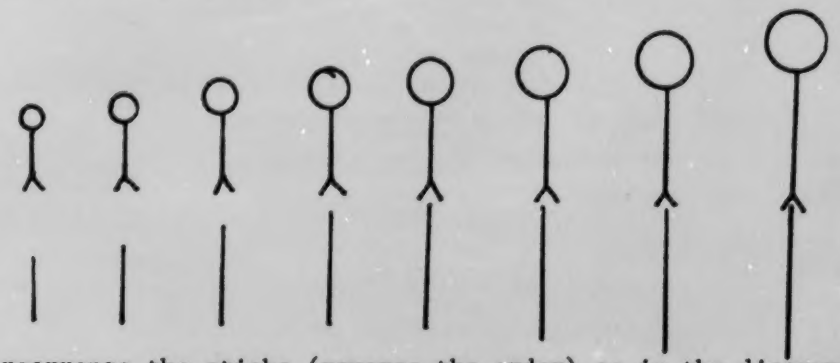
"Why did you do it that way?"

TASK 7

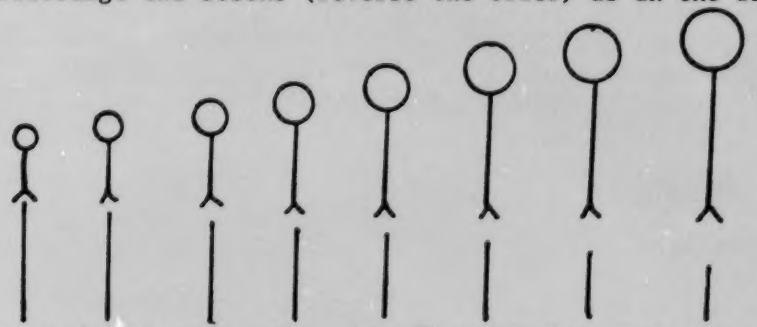
Cardinal-Ordinal number

Materials: 8 doll cutouts of different heights
8 straws of different lengths

Directions: Give the doll cutouts to the child and ask him to place them in order from shortest to tallest. After this has been done, say "Pretend the dolls are going for a walk and the straws are their walking sticks. Can you find which stick each doll would take?" After the child has arranged them as below



rearrange the sticks (reverse the order) as in the diagram below.



Pick a doll at random. Ask, "Which stick goes with this doll?"

TASK 8

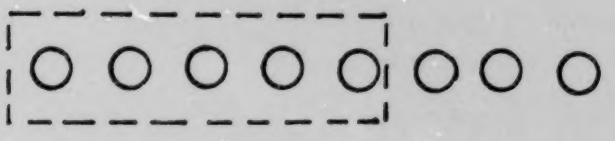
Cardinal-Ordinal number

Materials: 8 counters
a piece of cloth

Directions: Line the 8 counters up as follows; now beginning with the

leftmost one above, say, "If this is the first one, what is this one (pointing to the next one), what is this one" (pointing to the third one), etc. Continue until all counters have been named and the last one called eight.

Now cover the first five with the cloth as shown below.



Say, (pointing to the eight one), "Remember you said this was the eight one, can you tell me how many are covered?"

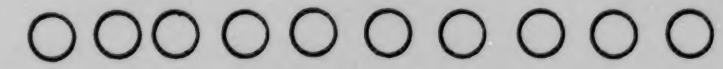
Variation: Say "number one," "number two," etc. instead of first, second, third, etc.

TASK 9

Cardinal-Ordinal number

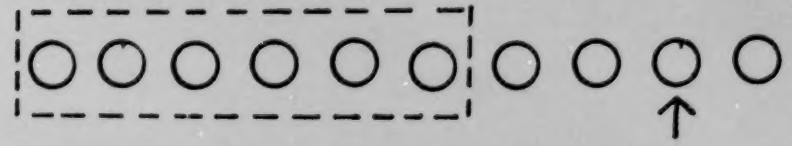
Materials: 10 counters
a piece of cloth

Directions: Place the counters on the table before the child as follows:



Now cover the first 6 counters as shown below.

Now say, "I have the first six counters covered. Which one is this (pointing to the ninth one)?"



Variations: Cover different numbers of counters. Say, "I have number one, number two, number three, etc., covered. How many do I have in all?"

TASK 10

Conservation of numerical equality (sequence of four tasks) (first of four)

Materials: 10 orange sticks
10 blue sticks

Directions: Say the following to the student, "I am going to count out 10 blue sticks and place them in a straight row. Each time I take a blue stick, you place an orange one beneath mine. Let me do the counting. You just place your stick below mine each time. Ready, begin. I take one, you take one, I take one, you take one, etc." The final arrangement should look as follows:

Blue sticks	B	B	B	B	B	B	B	B	B	B
Orange sticks	O	O	O	O	O	O	O	O	O	O

Now ask the following questions:

1. Are there more orange sticks or more blue sticks? How do you know?

2. If I push my blue sticks into a bunch, will there be more orange sticks or more blue sticks? (Teacher pushes the blue sticks together) How do you know?

TASK 11

Conservation of numerical equality (second of four)

Materials: 18 orange sticks
18 blue sticks
a piece of paper

Directions to the student: As I put down a blue stick, you put down an orange stick below mine. We are going to make 3 groups of six.

Group 1	Group 2	Group 3	
BBBBBB	BBBBBB	BBBBBB	B = blue stick
OOOOOO	OOOOOO	OOOOOO	O = orange stick

Now ask the following questions:

1. In group 1, are there more orange sticks or blue sticks? How do you know?
2. In group 2, are there more orange sticks or more blue sticks? How do you know?
3. In group 3, are there more orange sticks or blue sticks? How do you know?
4. Altogether in the three groups, are there more orange sticks or more blue sticks? How do you know?
5. (The instructor covers all the orange sticks with a piece of paper so that they remain untouched, but also unseen). Are there more orange sticks or blue sticks?
6. (The instructor bunches the blue sticks of group 1 together) Are there more orange sticks or blue sticks? How do you know?
7. (The instructor bunches the blue sticks of group 2 together) Are there more orange sticks or blue sticks? How do you know?
8. (The instructor bunches the blue sticks of group 3 together) Are there more orange sticks or blue sticks? How do you know?

TASK 12

Conservation of numerical equality (third of four)

Materials: 18 orange sticks
18 blue sticks
3 rubber bands

Directions to the student: We are again going to count out three groups of six sticks. Remember, each time I take a blue stick, you take an orange one and lay it below mine. (See task 11)

Now ask the following questions:

1. If I put an elastic band around my blue sticks in group 1, would there be more orange sticks or more blue sticks? (instructor does the bundling) How do you know?
2. If I do the same to my blue sticks in group 2, would there be more orange sticks or blue sticks? How do you know?
3. If I do the same to my sticks in group 3, would there be more orange sticks or blue sticks? How do you know?
4. I am going to take my three bundles and put them end to end to make a train. You do the same to your loose orange sticks. (When this has been done, ask) Are there more orange sticks or blue sticks? How do you know?

TASK 13

Conservation of numerical equality (fourth of four)

Materials: 18 orange sticks
18 blue sticks

Directions to the student: Everytime I put out a blue stick, you put out an orange one so that we have 2 groups of six.

Now ask the following questions:

1. Are there more orange sticks or blue sticks?
2. If I bundle a group of 6 together and leave the others loose like your orange sticks, would there be more orange sticks or more blue sticks? How do you know?
3. Let's put out a third group of 6. Now, are there more orange sticks or blue sticks? How do you know?
(At this point, your arrangement should look like this.)

BBB		
BBB	BBBBBB	BBBBBB
OOOOOO	OOOOOO	OOOOOO

4. How many blue sticks do I have? How do you know?
5. Instead of saying that I have 18 loose sticks like you have, can you rename or call my sticks something else? (1 bundle and 12 loose)
6. How else can I rename the number 18? Can I make anymore equal groups? (2 bundles 6 loose)
7. Is there another way? (3 bundles 0 loose)
8. How many sticks do we put into bundles when we do our work during math time? (10)
9. How would we name our 18 sticks as we do during math time? (1 bundle of 10 and 8 loose)

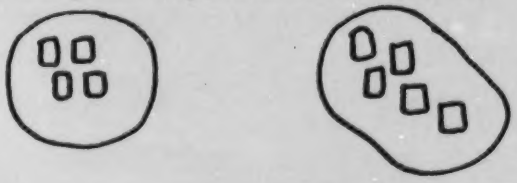
TASK 14

Conservation of quantity

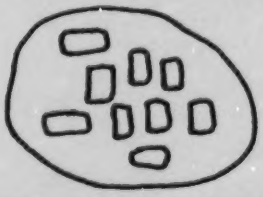
Meaning of addition

Materials: Collection of counters

Directions: Tell the child to count out a pile of four counters. Now have him count out five more. Keep the two piles separated on the table.



Ask the child to watch as you push the two groups together into one pile.



Now ask, "Do we have more counters in this one pile than we had in the two piles?"
"Do we have fewer counters in this one pile than we had in the two piles?"
"Do we have the same amount of counters in this one pile as we had in the two piles?"

Variations: Use candy instead of counters

TASK 15

Conservation of quantity

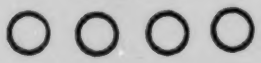
Meaning of many names for a number

Materials: 8 counters

Directions: Pretend the counters are candies. How many pieces of candy do you have? Now pretend that yesterday you were very helpful to your parents and in reward they gave you these candies. Suppose they gave you this many in the morning (make a pile of four) and this many in the afternoon (make a pile of four).

yesterday

Morning

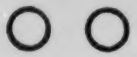


Afternoon

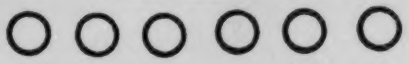
Now suppose that today you are also helpful and also will receive some candies. However, today you aren't very hungry in the morning and only take this many (make a pile of two) but in the afternoon you take this many (make a pile of six)

today

Morning



Afternoon



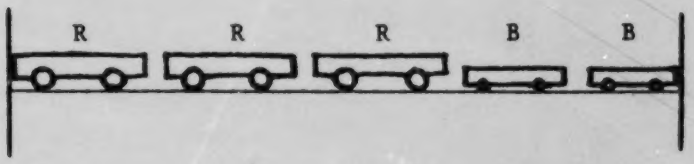
Now ask, "On which day would you have more candy to eat, yesterday or today?"

TASK 16

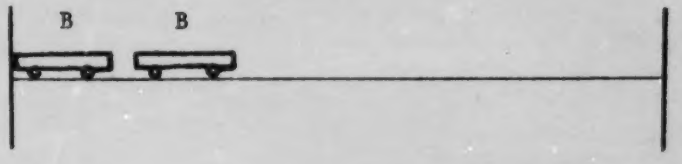
Meaning of commutativity

Materials: 3 red cars
2 blue cars (blue cars are smaller than the red ones
2 sticks to use as markers
piece of construction paper

Directions: Make a straight line on the construction paper. Now put the cars on the line, begin with the red ones and follow with the blue ones. Place the sticks at each end of the row of cars as in the diagram below.



Call attention to the fact that all cars are parked between the two markers. Now remove the cars (leave markers). Begin again by placing the two blue cars. Stop after the blue cars are placed and ask, "Is there enough space for the red cars?" "Why?"

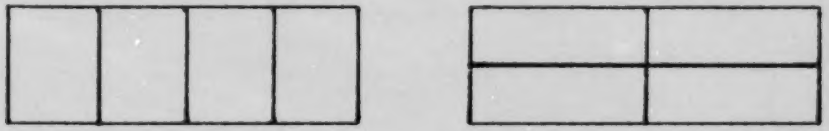


TASK 17

Meaning of a fraction

Materials: Two 4" x 8" congruent rectangular regions

Directions: Take the two regions and ask the child if they are the same size. The child should be allowed to place one on top of the other to verify. Now mark region A and region B as in the diagram below.



A B

Ask, "What is a fraction name for each part of region A?"
"What is a fraction name for each part of region B?"

Now point to a part of region A and ask if that part is the same size as one part of region B (pointing to a part of B).

144

143

TASK 18

Concept of a fraction

Materials: Modeling clay
plastic knife
dolls or doll cutouts

Directions: Use the modeling clay to represent a cake. Ask the child to divide the cake between two dolls so each has the same amount.

Repeat with 3 dolls, 4 dolls, 5 dolls, etc. After the dividing, ask if the parts were put back together, would we have the whole cake?

Variation: Child has only to show how he would divide the cake.

PART III

ERROR ANALYSIS

Chapter 10

ERROR ANALYSIS: AN ITEM ANALYTIC INVESTIGATION OF DIAGNOSTIC TESTS

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University of Dallas
and
Lyndal R. Hutcherson
Carrollton-Farmers Branch
Independent School District

The primary objective of this study was to perform a district-wide analysis of common errors in computation among third, fifth, and seventh graders. The Carrollton-Farmers Branch Independent School District wanted objective data on the current status of mathematics performance to plan an effective program for the future. Evaluation had previously been based on teacher and principal opinions along with standardized achievement test data. This study was undertaken to aid the district in their planning.

A second objective was to isolate the errors made by the students who performed at a below average level. Those errors were compared to the errors of the students who performed well. This study was based on recent studies (Ashlock, 1976; Hutcherson, 1975) which have shed light on the type of common errors students are likely to make. The occurrence frequency in five specific error classifications was to be studied: (1) correct fact, wrong operation; (2) basic fact error; (3) concept error; (4) place value errors; and (5) miscellaneous errors. A search of available tests revealed no test suited to the objectives. New tests were developed to diagnose common errors over specific content and yield data useful in planning remedial work for individuals.

PROCEDURES AND DATA ANALYSIS

The subjects in this study were 224 third, 254 fifth and 122 seventh graders. Each elementary building principal selected one third and one fifth grade class which he felt would reflect a cross section of abilities within the entire class. The seventh grade classes were two each of the basic, regular and honors sections. Hence, the sample was believed to represent a cross section of socio-economic backgrounds as well as ability levels.

The three group diagnostic inventories, forms A, B and C were developed and validated during the 1976-77 school year. The content was determined by the district-wide objectives, but at the same time limited to the four basic operations on whole numbers, fractions, decimals, and percents. Each inventory was designed to test computational skills at two or more grade levels. Form A was administered to third graders and tested skills usually introduced

at grade levels two and three. Form B was administered to fifth graders and tested skills usually introduced at grades four and five. Form C was administered to seventh graders and tested skills usually introduced at grades five, six, and seven. The three tests were further validated by asking a committee of experienced math educators to verify that the inventories were appropriate for the grade level.

Individual multiple choice items with four alternatives were constructed for each form using common errors as the three distractors. For example, 51 on the grade three inventory had 21 as the correct answer. The -30
three distractors, 81, 20 and 80 represented the common errors:
wrong operation, basic fact, and a miscellaneous error.

A fall pilot study and pretest yielded two important types of information: a base line of student achievement which both teachers and administrators could use in planning instruction, and student response data for item analysis of the tests.

Each item of each inventory was analyzed using computer programs developed by Grady (1976). Each item yielded an index of discrimination, an index of difficulty and upper and lower group data for distractor analyses (Sax, 1974). Items with low or negative indices of discrimination were improved or replaced. The item analyses revealed which distractors, common errors, were selected by the three groups: upper group consisting of students who did well overall, lower groups, consisting of students who did badly overall, and all students together. The frequency of selection of particular errors by the three groups revealed patterns of learning deficiencies useful in planning and remediating instruction. For example, if third graders who did well overall made frequent regrouping errors, then the district could make plans to improve or modify this aspect of the second and third grade curriculum. However, if only the lower group was experiencing a particular error pattern, then instruction could be altered by just those students. The item analysis also yielded individual diagnostic profiles for the classroom teachers to use in remediation. Reliability data was gathered from the Fall data. KR_{21} statistics for the three tests were: A = .92, B = .91 and C = .93.

Revised tests were then readministered in the Spring to the same sample. Working with the base data gathered in the Fall and the frequency of common errors in the different content areas, the data were compared for improvement in error frequency and observed for changes in the error patterns. Chi-square statistics were used to test for statistically significant differences in the frequency distribution of errors from the Fall to the Spring.

RESULTS AND DISCUSSION

For each of the three forms of the Inventory, it was discovered that a sixth category, omissions, was needed. This category was informative in that many of the students left problems out on the pretest in the Fall.

The data for the third grade are summarized in Tables 1 and 2. As both tables show, the number of errors made by those students in the lower one-third

of the subjects was nearly the same as the upper and middle one-third combined. The total of 3666 errors in the Fall testing was very high. The mean number of errors on the pretest was 16.37 errors per student on a test of forty items.

Table 1

Number of Errors by Category
Third Grade Summary

CATEGORY	Upper 1/3		Middle 1/3		Low 1/3		All	
	PRE	POST	PRE	POST	PRE	POST	PRE	POST
Basic Fact	67	15	93	69	229	154	389	238
Concept	118	11	103	26	156	129	377	166
Wrong Operation	103	18	112	53	107	151	422	322
Place Value	126	26	193	87	307	228	626	341
Miscellaneous	179	39	260	92	559	243	998	369
Omissions	243	11	216	29	395	233	854	273
TOTAL	836		977	356	1853	1238	3666	1709
X^2	29.17*		56.94*		80.13*		106.17*	

*Significant at .01 level

The distribution of errors on the posttest is given in Table 1, also. On the posttest, a total of 1709 errors occurred, with an average of 7.91 errors per student. The number of errors on the posttest for the upper one-third of the students represents an 85% decrease in the number of errors. The chi-square statistic indicated a significant change in the distribution of errors from the pre- to posttest. The middle one-third of the students had a 64% decrease in the number of errors. This change was also significant. The categories contributing to the change were basic fact and omissions. The lower one-third of the students had a significant decrease in the number of errors from the pre- to posttest, also. However, the number of errors on the posttest was still high; an average of 16.5 errors per student. The average number of errors per student on the pretest was 24.7. For the lower one-third of the students, there was little change in any of the areas except wrong operation and miscellaneous.

The entire third grade test consisted of whole number operations categorized into the four basic operations: addition, subtraction, multiplication and division. The large number of errors in multiplication and division were expected in that very few beginning third grade students have been exposed to these operations. On the pretest, subtraction accounted for 31% of the total number of errors, while on the posttest, subtraction accounted for 40% of the total. The beggest decrease in the number of errors was in multiplication. the problems in this area were simple basic facts. Many of the students in the lower one-third of the achievers were not exposed to either multiplication or division during the year.

Table 2 gives the distribution of errors in the four operations on both the pre- and posttest for all groups of students.

Table 2

Number of Errors by Operation
Third Grade

OPERATIONS	Upper 1/3		Middle 1/3		Low 1/3		All	
	PRE	POST	PRE	POST	PRE	POST	PRE	POST
Addition	102	27	203	79	459	288	764	393
Subtraction	266	46	264	134	611	506	1141	683
Multiplication	124	18	239	39	429	184	791	240
Division	354	29	273	104	354	260	970	393
X^2	16.65*		33.06*		40.66*		69.35*	

*Significant at .01 level

As evidenced by the chi-square statistics, significant changes in the distributions occurred at all three levels as well as for all students at the third grade level. Further study revealed that the majority of the popular incorrect responses were caused by inadequate knowledge of both place value and the concept of subtraction.

For the fifth grade, the large number of errors on the pretest, (see Table 3) was probably due to the fact that the students had not been exposed to much of the material covered on the second part of the test.

Table 3

Number of Errors by Category
Fifth Grade Summary

CATEGORY	Upper 1/3		Middle 1/3		Low 1/3		All	
	PRE	POST	PRE	POST	PRE	POST	PRE	POST
Basic Fact	50	8	87	21	81	83	218	112
Concept	345	73	355	190	471	421	161	149
Wrong Operation	20	19	48	75	93	55	851	716
Place Value	162	86	273	177	416	453	1171	684
Miscellaneous	230	72	233	21	320	316	783	409
Omissions	158	24	264	108	581	188	1003	320
X^2	57.50*		136.77*		163.70*		165.51*	

*Significant at .01 level

It should be noted that the number of omissions in the lower one-third of the students was greater than both the upper and middle thirds together. The number of errors in the wrong operation category for the lower students was also greater than the sum of the errors in the other two groups of students. The number of place value errors for the lower one-third of the students decreased by 71% while the decrease in the middle one-third was 58%. The decrease in the lower one-third was only 23% and the change in the omissions category accounted for 20% of the total change for the lower students. There was very little change in the other five categories. There was an overall decrease of 43% for all the students combined at the fifth grade level. The mean number of errors on the pretest was 16.48 and the mean on the posttest was 9.41. The changes accounting for the statistical significance were place value and omissions.

Table 4 gives the distribution of errors in the content areas for the pre- and posttest.

Table 4

NUMBER OF ERRORS BY OPERATION
FIFTH GRADE SUMMARY

OPERATION	PRETEST	POSTTEST	X^2
Whole No. Addition	230	130	
Whole No. Subtraction	200	123	
Whole No. Multiplication	670	305	
Whole No. Division	1095	539	
Decimal Operations	331	215	
Fraction Operations	1068	542	
TOTAL	3594	1854	14.01*

*Significant at .05 level

The greatest change occurred in the categories of whole number multiplication, whole number division, and all fraction operations. Seventy-five percent of the errors on the posttest occurred on these three categories.

For the seventh grade, Table 5 gives the distribution of error types for both the pre- and posttests.

Table 5

NUMBER OF ERRORS BY CATEGORY
SEVENTH GRADE SUMMARY

CATEGORY	PRETEST	POSTTEST	X^2
Place Value	764	578	
Wrong Operation	230	171	
Basic Fact	67	45	
Concept	1129	735	
Miscellaneous	324	715	
TOTAL	2512	2244	3021.11*

*Significant at .01 level

There was a statistically significant drop in the number of concept errors from the Fall testing to the Spring. There was also a significant difference in the distribution of errors across content areas as shown by the data in Table 6.

Table 6

NUMBER OF ERRORS BY TOPIC
SEVENTH GRADE SUMMARY

TOPIC	Upper 1/3		Middle 1/3		Lower 1/3		All	
	PRE	POST	PRE	POST	PRE	POST	PRE	POST
Whole Numbers	47	30	350	93	290	243	687	366
Fractions	225	100	830	452	538	480	1593	1032
Decimal	90	37	248	123	168	125	506	285
Percent	74	51	136	110	92	100	302	261
Miscellaneous	87	67	198	138	148	128	433	323
X^2	8.14		54.29*		4.49		78.21*	

*Significant at .01 level

There was a significant drop in the number of errors in the categories of whole numbers, fractions and decimals. The decrease in the number of errors in the other two categories was not significant. Those students scoring in the lower one-third of the population did not show a significant change in their distribution of errors. There was also little change in their number of errors in each category. In the middle group, there was a significant difference in the distribution of errors from the pre- to posttest. The number of errors on the first three categories on the posttest was approximately one

half the number of errors on the pretest. In the categories of percent and miscellaneous, there was a decrease in the number of errors; however, the amount of change was not as much as occurred in the first three categories.

For the upper group, even though there was a decrease in the number of errors in every category, there was not a significant change in the distribution. The biggest change came in the categories of fractions and decimals. Again, as in the middle one-third, the amount of change in percent and miscellaneous was much lower than in the other three categories.

SUMMARY AND CONCLUSIONS

Among third graders, item analyses indicated errors in the main error categories were reduced during the year for average and above average students but only slightly reduced for the below average students. Lower ability students were experiencing errors in subtraction and division. Place value errors occurred in significant frequencies in both the average and below average groups.

Among fifth graders, the decrease in the number of concept and wrong operation errors was very small. Significant frequencies of errors also occurred in whole number division and all fraction operations. This was a disappointing finding in light of the fact that so much of the fifth grade curriculum is focused on these two topics.

Among seventh graders, achievement gains were minimal. Item analysis indicated a significant change in the distribution of errors in the general categories. Among lower ability students, the number of errors was not reduced in any of the specific content areas. The average students' errors were reduced for whole numbers only.

This study established that a district can perform relevant group and individual diagnosis of computational error patterns using multiple choice tests and computerized data analyses. Through the procedures employed in this study, individual teachers were furnished diagnostic information about their individual students and the district gained important diagnostic and assessment data useful in planning curriculum changes and teacher inservice.

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Chapter 11

THE ERRORS CHILDREN MAKE: ARE THEY MATHEMATICAL OR CLERICAL?

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University of Florida

This investigation sought information relative to clerical skills associated with test-taking; an ability which many consider to be central to test performance. In particular, clerical skills needed for proper completion of separate answer sheets used in conjunction with computer-scanning equipment have often been listed as a criticism of standardized tests (Harneschfeger and Wiley, 1976; Houts, 1977; McKenna, 1977). Data to support this criticism has not been referred to by these critics, thus leaving us to appeal only to personal experiences. The goal of this investigation was to determine to what extent clerical skills affect test scores and to identify particular test features which require the use of clerical skills. The results of such an investigation should be useful to those who construct tests and to those who are responsible for the interpretation of test results.

BASIC QUESTIONS

1. Given the answers to multiple-choice test items, can students find the correct choice from among five options and transfer the correct choice to a standard computer-scan answer sheet?

2. Given simple, one-digit addition and subtraction problems, can students find the correct answers from among five options and mark the correct choice on a computer-scan answer sheet?

3. Does skipping items on a test cause any difficulty in completing a computer-scan answer sheet correctly?

4. Does having numerical answers the same as numerical choices on multiple-choice test items cause difficulty in student performance?

5. What differences occur in student performance on a multiple-choice test when using a computer-scan answer sheet as opposed to marking answers directly on the test booklet?

DESIGN

Twenty-nine teachers from seven schools volunteered to have their classes serve in the investigation. Thirty-seven classes were randomly

selected which led to the identification of 842 students in mathematics classes (6th, 7th, 8th grades, high school general math and Algebra I) who provided data for questions 1 and 2, above. A subsample of 719 students of the original 842 provided data for questions 3 through 5.

During the Spring of 1977, an investigator-developed test was administered to the 842 students by their regular mathematics teacher. One week later, 719 of the same sample took the same test again. The difference in the two test administrations was the use of a standard, commercially available computer-scan answer sheet for the first administration. For the second administration, subjects circled their choices directly on the test.

The 33-item test used for both administrations contained two parts: items in which a number was given for which the student was to identify the same number from among 5 choices; and items in which a one-digit addition or subtraction problem was presented for which the student was to solve and select the correct response from among five choices. Part I contained twenty items; 9 in which the number presented was a 2-digit number; 9 in which the number to be matched was a one-digit number and 2 items for which no number was presented but for which the statement "SKIP THIS ONE" was presented. Part II began with item number 23 thus forcing the skipping of items 21 and 22, then eleven addition and subtraction problems for which the answer was a one-digit number between 1 and 5 were presented.

Specific, written instructions were read by the teacher. The students were informed only that the test was supposed to find out how quickly they could match numbers and do simple addition and subtraction problems. The instructions also apprised students that items 21 and 22 were missing. Each class was given a full ten minutes to complete the test and were allowed to go back over the test if time remained.

For the first administration, the "General Purpose Answer Sheet" produced by National Computer Systems was used for recording responses. Spaces for item responses were printed in columns of 10 each. Five numbered circles (1 through 5) were presented for each item.

RESULTS

Mean performance by the entire sample on each portion of the test and on the total test is presented in Table 1. The overall mean performance on Part I was 19.3 out of a maximum score of 20. This performance does not indicate major difficulty or significant failure on the part of the Ss.

Table 1

Mean Performance on Parts I, II and
Total Test with Answer Sheets, by Grade Level

PART I	n	$\bar{X}$	s.d.	% non-perfect
6th	175	19.1	2.9	24.6
7th	181	19.4	1.8	22.1
8th	212	19.3	2.3	26.4
Gen. Math	156	19.1	2.9	25.6
Algebra I	118	19.9	0.4	6.8
TOTAL	842	19.3	2.9	22.2
PART II (13 max.)				
6th		11.9	2.2	40.0
7th		12.3	1.9	30.4
8th		11.9	2.3	42.9
Gen. Math		11.5	2.7	51.3
Algebra I		12.4	1.7	27.1
TOTAL		11.9	2.2	39.0
TOTAL TEST	range			
6th	5-33	30.9	4.1	50.3
7th	7-33	31.8	3.3	42.4
8th	4-33	31.2	4.0	53.3
Gen. Math	4-33	30.6	5.2	59.0
Algebra I	22-33	32.3	1.7	32.2
TOTAL		3.13	4.1	48.2

But the results expressed as the percentage of Ss who did not achieve a perfect score holds more meaning for this investigation. For the entire sample, 22.2% missed one or more items in which the number to be found from among the choices, was given. The answer, therefore, to Question 1 is approximately one-fourth of the Ss were not able to find and transfer the correct choice to a computer-scan answer sheet.

Part II of the test provided data relevant to Question 2. From Table 1, the mean performance of the entire sample on Part II was 11.9 with a possible maximum score of 13. Again, the percentage of Ss who missed one or more items is pertinent to answering the question. Given simple, one-digit addition and subtraction problems, 39.0% of the Ss were not able to select the correct answer and mark the correct choice number on a computer-scan answer sheet.

Analysis of the types of errors Ss made on the total test is important for determining the answers to Questions 3 and 4. Results of this analysis are presented in Table 2. Six error types were found which accounted for 777 out of the total 1255 errors made. From this data, the two categories

Table 2

Analysis of Errors for Both Administrations
(WITH/WITHOUT Answer Sheets)

Error Type	Frequency		% of total errors		% of errors classified	
	WITH	WITHOUT	WITH	WITHOUT	WITH	WITHOUT
1. Filled spaces to be left blank	311	--	25%	--	40%	--
2. Started in wrong place or side	165	--	13%	--	21%	--
3. Gave answer, not choice number	161	6	13%	9%	21%	15%
4. Wrong operation	63	23	5%	35%	8%	58%
5. Fact error	23	11	9%	17%	3%	27%
6. Phase shift	54	--	4%	--	7	--
7. Total errors classified	777	40	62%	61%	--	--
Total errors made	1255	66				

of errors "Filled spaces to be left blank" and "Phase shift" relate directly to the items which read "SKIP THIS ONE." An error was classified as "Filled spaces to be left blank" if the subject had marked a response for one of the four items which were to be skipped. In some instances, not skipping items that were to be skipped led to errors on subsequent items. The analysis found that the sequence of responses was correct but the subject had marked the responses on the computer-scan answer sheet one or more item response spaces out-of-phase. These errors were classified as "Phase shift" errors. The total of 365 errors assigned to these two categories constituted 47% of the 777 errors which were classified 29% of all 1255 errors made. These results indicate that skipping items caused major difficulty for these Ss in correctly completing a computer-scan answer sheet.

The category "Gave answer, not choice number" in Table 2, refers to errors in which the response made was either the number to be matched (Part I) or the correct numerical answer to the addition or subtraction problem (Part II) rather than the multiple choice number designating the

match or correct arithmetic answer. The 161 errors of this type constituted 21% of the errors which were classified and 13% of all errors made on the total test. Question 4 then must be answered affirmatively, having numerical answers the same as the numerical choices did cause difficulty.

Table 2 provides comparative data for the subsample of 722 Ss who were readministered the test but without using the computer-scan answer sheet, two weeks after the first administration. Relative to error types found when Ss marked their responses directly on the test booklet, the categories "Filled spaces to be left blank," "Started in the wrong place or side," and "Phase shift" could not occur without the answer sheet. Since these error types accounted for a majority of the errors classified for the first administration, it is natural that fewer errors were made for the second administration. Unfortunately, the analysis does not identify the source of the reduction of 478 errors not classified on the first administration to the 26 errors not classified when Ss marked their responses directly on the test booklet.

Table 3 presents the mean performance of the 722 Ss by grade level for each test administration. Table 4 presents the results of computation of correlation coefficients and t-values for total test scores of Ss for each administration. For the entire sample, a weak, significant correlation ($r = .21$) exists indicating a slight positive correlation between the score received on the test when a computer-scan answer sheet was used and the score received when answering directly on the test booklet. Due to the simplicity of the content of the test, this low correlation suggests that reliability of the test results is affected by the use of a computer-scan answer sheet. The significant t-values for each grade level and the total sample indicate that Ss scored significantly higher on the test when answering directly on the test booklet.

Table 3

Mean Performance of Subjects
Receiving Both Test Administrations

Group	n	With Answer Sheet $\bar{X}$	With Answer Sheet s.d.	Without Answer Sheet $\bar{X}$	Without Answer Sheet s.d.
6th Grade	164	30.93	4.7	32.37	3.3
7th Grade	159	31.69	3.5	32.79	1.7
8th Grade	149	31.13	4.1	32.62	2.3
General Math	138	30.46	5.4	32.36	3.3
Algebra I	109	32.39	1.8	32.96	0.2
TOTAL	719	31.22	4.3	32.60	2.5

Table 4

Correlation Coefficients and T-Values
for Both Administrations

Group	r	t-value
6th Grade	.18	-3.50*
7th Grade	.54*	-4.65*
8th Grade	.07	-3.99*
General Math	.21*	-3.92*
Algebra I	.03	-3.98*
TOTAL	.21*	-8.34*

*p < .05

Table 5 provides additional data for answering the questions regarding the differences that resulted from the two test administrations (Question 5).

Table 5

Differences Between Test Administrations
(With/Without Answer Sheet)

A. Missed one or more:

	WITH	WITHOUT
6th	81 of 164 (49%)	21 of 164 (13%)
7th	65 of 159 (41%)	11 of 159 (7%)
8th	60 of 149 (54%)	16 of 149 (11%)
General Math	83 of 138 (60%)	18 of 138 (13%)
Algebra I	33 of 109 (30%)	3 of 109 (3%)
TOTAL	342 of 719 (48%)	69 of 719 (10%)

B. Change from "WITH" to "WITHOUT:" (frequency of subjects)

	HIGHER	SAME	LOWER
6th	76	77	11
7th	63	90	6
8th	72	72	5
General Math	78	56	4
Algebra	32	75	2
TOTAL	321	370	28

(Continued)

C. Frequencies of those not perfect "WITH" getting perfect on "WITHOUT:"

	Not perfect "WITH"	Perfect "WITHOUT"
6th	81	71 (88%)
7th	65	60 (92%)
8th	80	66 (83%)
General Math	83	68 (82%)
Algebra I	33	31 (94%)
TOTAL	342	296 (87%)

Considering the expectation of a perfect score on a test such as the one administered, one or more items wrong was considered to be a significant measure of performance. Part A of Table 5 provides the number of Ss who missed one or more items on either test administration. It was notable that 342 Ss missed one or more items when using the computer-scan answer sheet and 69 Ss missed one or more items when marking responses directly on the test booklet.

Part B of Table 5 indicates that not all students scored higher when answering directly on the test booklet. A total of 28 Ss scored lower without the use of the computer-scan answer sheet. It was also the case that some of the Ss who scored the same or higher on the second test administration still did not receive a perfect score. This data simply indicates the relative change from the first to the second test administration.

Part C of Table 5 does show that of the 342 Ss who did not respond perfectly when using the computer-scan answer sheet, 296 (87%) of them did receive a perfect score when answering directly on the test booklet.

DISCUSSION

This investigation sought to determine whether or not clerical errors associated with test taking are made by secondary school students, and if so, to determine the types of clerical errors present. It was found that roughly half of the students involved in the investigation made at least one error on a test for which they should have had a perfect score. Subsequent analysis of the errors led to the identification of factors associated with correctly completing a computer-scan answer sheet as the major source of clerical errors. It is to be noted that confusing the numerical answer to an item with the choice number was a clerical error identified for both the administration with an answer sheet and the administration without an answer sheet. At least half of the 1255 errors made on the first administration were clerical errors. Clerical errors may have been the source of an additional 478 errors made, but the analysis performed did not allow for classification of these errors.

To researchers in mathematics education, these findings suggest that for investigations in which a computer-scan answer sheet is used in

collecting data, it would be good to attempt to ascertain some measure of the clerical errors made by the sample involved. This investigation implies that the validity and reliability of an instrument are both affected by the use of a computer-scan answer sheet. It would, therefore, be in the best interests of the researcher to collect data without the use of computer-scan answer sheets, whenever possible.

To educators whose job it is to interpret test scores, these results simply verify something which has been thought to be true. The results do indicate that clerical errors may be playing a larger part in the performance scores of students than we might have thought before.

In interpreting test scores, the results of this study suggest two things. First, students who obtain high scores on tests which use computer-scan answer sheets deserve every point that they get. That is, for these students who are exhibiting both clerical skills as well as skills and knowledge of the content tested, we can feel fairly comfortable that the scores are valid and reliable. Second, students who do not score as well as expected on a mathematics test would not necessarily score higher if a computer-scan answer sheet had not been used. Not knowing whether the low score is due to a lack of clerical skills or due to a lack of skill and knowledge in the mathematical content being tested suggests that we need alternate methods for measuring the content skills.

Upon learning the results of this investigation, two responses will be typical. There are those who will claim "Who cares?," suggesting that students should be able to correctly complete a computer-scan answer sheet and if they cannot, then they have no business passing the test anyway. This is obviously the "head-in-the-sand" response, but one with which this investigator has already been confronted. Certainly if this were correct, then we would not have some grade levels taking standardized tests with a computer-scan answer sheet and other grade levels answering directly on the test booklet. Hopefully, we in mathematics education will be more interested in obtaining valid and reliable information regarding the mathematical capabilities of students, and thus, will not take this response position.

Others will offer suggestions intended to reduce the number of clerical errors students make when responding on computer-scan answer sheets. Suggestions will include that answer sheets use letter choice instead of number choices, or use of staggered choice designators having the even and odd-numbered items presenting different choices. This will of course, eliminate the confusion of the numerical answer with the numerical choice designator but based on the results of this study, it is doubtful that skipping errors will be eliminated through these modifications. It would certainly be appropriate to investigate these modifications empirically.

Another well-intentioned suggestion will be to plan and conduct special instructional programs intended to teach children to make fewer clerical errors when completing a computer-scan answer sheet. The unfortunate thing about this suggestion is that instead of eliminating the source of the problem, we would fall into the common educational practice of trying to patch up a less than desirable practice by addition to an already crowded curriculum.

Consider how powerful such a training program would have to be in view of the number of clerical errors that were made in this investigation when a setting of low anxiety was present for testing overly simplified performance. No, there is not much hope for a training program.

Both response groups miss the obvious solution, in the opinion of this investigator. We should simply have students respond to test items by writing directly in the test booklet. This practice had been used before the advent of computer-scan answer sheets. Arguments against returning to this practice will more than likely be based upon the expense of scoring tests taken in this manner. But considering the waste of money by obtaining questionable results on tests when a computer-scan answer sheet is used, at least the expense would provide more reliable information.

It is important to note that this investigation nor the discussion of its findings is meant to suggest that children are performing in mathematics as well as we would wish. Children's difficulties in mathematics are evident without the information provided by tests which use computer-scan answer sheets. We must continue to strive to assist students in the achievement of mathematical knowledge and skill but for the sake of the individual, not the sake of a test score.

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Chapter 12

A SOVIET VIEW OF THE FORMATION OF GEOMETRY CONCEPTS IN NORMAL AND MENTALLY RETARDED STUDENTS

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Three research reports form the data base for the discussion that follows. These are articles by Tishin (1952/1975) and Khanutina (1957/1975) and a book by Krutetskii (1968/1976).[1] Summaries of the relevant parts of these works are provided in Appendix A through Appendix F. The most important conclusions are summarized below:

1. Retarded pupils are successful in distinguishing geometric figures (two-dimensional) which can be defined by a single essential characteristic. They are not successful in distinguishing figures which can be defined only by several essential characteristics.

2. Modeling or selecting a geometric form (two-or three-dimensional) is significantly affected by the pupil's facility with language.

3. Success in providing justification of choices of models of geometric forms is dependent on whether the form is defined by only one or by several essential characteristics.

4. Geometric bodies (three-dimensional) are better understood than geometric figures.

5. Lack of understanding of the differences among the concepts of length, area, and volume is revealed in the use of incorrect symbolism.

6. Computing areas and volumes is easier when the dimensions are given than when dimensions must be measured. Computing areas and volumes is easier when the dimensions are given in whole numbers than when the dimensions are given in mixed numbers.

7. In comparing spatial features, pupils often focus on inessential characteristics.

8. Objects of similar dimensions are more difficult to distinguish than objects of quite different dimensions.

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[1]For these reports, the first date in the parentheses indicates when it was published in the Soviet Union. The second date indicates when it was published in the USA.

9. Structuring a problem is difficult.

10. Transferring solution techniques from one situation to another is difficult.

Tishin and Khanutina worked with defective (retarded) children. Krutetskii worked with incapable rather than retarded pupils. However, his results are similar. All the conclusions focus on the failure of the pupils to perform certain tasks. This negative approach makes it difficult to understand what actually characterizes retarded pupils. Surely, it is important to know what they can do as well as what they cannot do.

The differences in the behavior of the retarded and incapable pupils seem to be mainly in degree. Tishin and Khanutina used simpler tasks than Krutetskii. More sophisticated tasks used with the retarded pupils would likely have resulted in less success in completing the tasks.

It is difficult to use these results to draw precise implications for clinical work. However, the following appear to be important to the development of clinical procedures.

1. Any clinical procedure which requires students to write their responses should a priori define what degree of accuracy of symbolism is acceptable. Certainly, students who are referred to a mathematics clinic for remedial help are quite likely to use imprecise or inaccurate symbolism. Students in non-remedial clinical settings, however, may also be unable to use accurate symbolism.

2. Limits on acceptable verbalisms should also be set. Retarded and incapable students seem in varying degrees to be unable to express their ideas precisely and accurately. A clear understanding of what constitutes acceptable behavior seems essential.

3. Geometry concepts that are defined by more than one essential feature may elicit different behaviors than concepts defined by only one essential feature. For example, an investigation of the application of geometry concepts in problem solving settings might more clearly illustrate the processes of applying concepts if the concept to be applied is triangle rather than rectangle. The clinical procedure must be detailed enough to determine to what extent behavior is confounded by the difficulty of the concept.

4. Models of geometric forms might convey more information than the names of the same forms. The vocabulary of the interviewer is always of concern in a clinical investigation, but for investigations of geometric concepts, the investigator has the option of using models to accompany or to replace verbalism.

5. Any investigation of area, volume, or perimeter should include activities to determine whether the students focus on the correct

concept. Confusion among these concepts may be merely the association of the wrong concept to the label (name) being used. If so, it would seem reasonable that an interviewer could fairly easily correct the trouble. On the other hand, the confusion might be due to an inability to keep the three concepts separated. In that event, further interviewing might not be fruitful. The clinical procedure needs to provide enough information to distinguish these situations.

6. In a clinical investigation of spatial features, the relative sizes of objects may be a critical factor influencing the success of students. Serious attention needs to be given to the development of the materials.

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APPENDIX A: GEOMETRIC FORMS

Tishin, P. G. /Instructing auxiliary school pupils in visual geometry./ Proceedings of the Academy of Pedagogical Sciences of the RSFSR, 1952, 41, 81-164. (Soviet Studies in the Psychology of Learning and Teaching Mathematics, 1975, 10, 1-124).

1. Focus

The following ideas were investigated:

a. whether the pupils can draw geometric figures (two-dimensional) and model geometric bodies (three-dimensional);

b. whether they know the names of the geometric forms (both figures and bodies);

c. whether they can find the geometric form corresponding to a given name; and

d. whether they can find the name corresponding to a given geometric form.

2. Procedures

Subjects were six pupils at each of grades 1-7 in the auxiliary schools, which admit only retarded pupils, and pupils in grades 1-4 of the public schools, which admit normal pupils. Data were gathered through "individual investigation" (p. 4). Although, there were eleven series of investigations of geometric figures and eight series of investigations of geometric bodies.

3. Results and Conclusions

In the discussion below, P denotes procedure, R denotes result, and C denotes conclusion. All results and conclusions relate to the retarded pupils, unless otherwise indicated.

a. P: sketch or draw a figure given the name
R: The pupils drew figures in the following order of decreasing success: circle, triangle, square, rectangle.
C: Pupils confuse square and rectangle because these two figures are closely related. The circle and triangle are unrelated.

b. P: copy a figure from a model
R: The circle and triangle were copied precisely, but there was some confusion in copying the square and the rectangle.
C: The confusion results from the fact that there are no sharp differences between the square and the rectangle.

c. P: comparison of normal and retarded pupils
R: Normal pupils drew more precise figures. For both groups, however, the circle and the triangle were represented more accurately than the square and the rectangle.

d. P: model a geometric body in clay given the name
R: Almost all pupils modeled the sphere (ball). At grades 5-7, most pupils modeled the cube and the parallellepiped. Almost no one modeled the cylinder, cone, or pyramid.

e. P: copy a body in clay given a model of the body
R: This was a much easier task than making a model given only the name.

f. P: comparison of normal and retarded pupils
R: The normal pupils were more accurate than the retarded pupils.

For both groups, the bodies were copied, in order of decreasing success, as follows: sphere, cube, parallelepiped.

g. P: name geometric figure given a model
R: In order of decreasing success, the figures named were triangle, circle, square and rectangle. In all cases, only one essential feature of the figure was focused on; e.g., only the four angles of the square. When pupils were presented a series of rectangles which became less and less oblong, the difficulty of identifying the figure increased.

h. P: name geometric body given a model
R: Almost all pupils named the ball; most named the cube. Other geometric bodies; i.e., parallelepiped, cylinder, cone, pyramid; were named by words like jar, box, etc. The sole exception was that seventh graders named the parallelepiped correctly.

i. P: comparison of normal and retarded pupils
R: Normal pupils were more accurate. Both groups named unfamiliar objects with words like jar, box, etc. Geometric bodies were better understood than geometric figures.

j. P: choose geometric form (from set of models) given name
R: In increasing order of difficulty, geometric figures chosen correctly were circle, triangle, rectangle, square; and geometric bodies, were ball, cube, parallelepiped, cone, cylinder, pyramid.

k. P: students asked to justify choices in (j) above
R: The circle and triangle were easily justified. Justifications for rectangle and square focused on only one of the three essential features in each case. Justifying choices for bodies was more difficult.
C: Figures which have only one essential feature are easily justified. Figures or bodies which have more than one essential feature are not clearly justified.

l. P: comparison of normal and retarded pupils
R: By grade 4, normal pupils gave accurate justifications for choosing figures which have more than one essential feature.

m. P: choose geometric form (from set of models) given model
R: From sets of plastic figures of several colors, pupils selected models on the basis of color, dimensions, and placement (i.e., easiest to reach, apparently). When asked to select all figures matching the given model, the task was often incomplete. Circles and triangles were easier to select than rectangles and squares. Choosing geometric bodies was easier than geometric figures.

n. P: comparison of normal and retarded pupils
R: Normal pupils were more successful and provided better justifications. Color exerted less influence on normal pupils. Normal

pupils used more systematic search procedures.

o. P: paper construction of models of geometric bodies
R: Retarded pupils needed demonstration of the constructions. Normal pupils did not, except at grades 1 and 2. After a demonstration, all pupils performed the constructions.

4. Summary

"Summing up what has been stated above, we conclude that:

a. The pupils of the auxiliary school easily selected a form accordingly to a model. The increase in difficulty depended on the supplementary features. Upon complicating the assignment by asking the pupils to select figures without regard to color and dimensions, the pupils of the auxiliary school made mistakes. If the figures were sharply distinguished from each other as, for example, circle and triangle, then all the pupils selected them according to a model without mistakes. The pupils of the auxiliary school made mistakes among the similar geometric forms, such as a rectangle and a square.

b. The next stage of difficulty in the study of geometric forms is their selection according to name, the representation of geometric figures in a drawing, and the modeling of geometric bodies.

c. The pupils of the auxiliary school encountered more difficulties in naming geometric forms than did the pupils of the public school. At first, the child does not master geometric concepts through precise verbal formulation characterizing the geometric form, but by means of concrete comparison with some object familiar to him from his daily life. For example, the cylinder is a little jar, a pipe, a little barrel, and the like.

d. The pupils of the auxiliary school have diverse ideas about geometric forms. They have a better idea of the solid form than of the plane form. At the same time, they do not have equal ideas of solid form; they have a better idea of circle and triangle and a poorer idea of square and rectangle.

e. More is required of the auxiliary school pupils than of the pupils of the public school, especially in organized studies, since in the process of practical activity these pupils master abstract geometric concepts with greater difficulties. Hence, one must conclude that studying geometric material should occupy an especially prominent place in an arithmetic course.

f. The auxiliary school pupils are capable of mastering elementary geometric ideas and concepts." (pp. 32-33)

APPENDIX B: SQUARE MEASURE

Tishin, P. G. /Instructing auxiliary school pupils in visual geometry./ Proceedings of the Academy of Pedagogical Sciences of the RSFSR, 1952, 41, 81-164. (Soviet Studies in the Psychology of Learning and Teaching Mathematics, 1975, 10, 1-124).

1. Focus

To ascertain the knowledge, abilities and skills of the pupils of the auxiliary and public schools with respect to square measures and the solution of problems in calculating areas.

2. Procedures

Subjects were six pupils at each of grades 6 and 7 in auxiliary schools and grade 4 in public school. There were three sets of problems.

"As material for this investigation, we took three rectangles /set I/ and two squares /set II/ of different dimensions which were traced out on a paper.

The dimensions of the rectangles were as follows:
a. length 12 cm, width 2 cm;
b. length 12.5 cm, width 7 cm;
c. length 9.5 cm, width 8.5 cm.

The dimensions of the squares were:
a. 6 cm
b. 7.5 cm

The following problems /set III/ were proposed to the pupils:
a. 'What is the area of a rectangle whose length is 7 cm and whose width is 3 cm?'
b. 'What is the area of a square having a side of 8 cm?'
c. 'What is the area of a Pioneers' room if it is 7 m long and 6 m wide?'
d. 'What is the area of a three room apartment if the length of the largest room is 6 m and its width is 5 m, the length and width of the middle room are each 4 m, and the smallest is 3 m long and 2 m wide?'" (pp. 34-35)

3. Results and Conclusions

Set I

For problem Ia, pupils measured and computed correctly, but at grade 6, the solution was expressed incorrectly symbolically. At grade 7, four of the six pupils provided correct solutions.

For problem Ib, the pupils either rounded off the measurements or, if they measured correctly, they multiplied incorrectly. At grade

7, three pupils measured correctly and two multiplied correctly, but no one gave a correct solution symbolically.

For problem Ic, performance was quite similar to that for problem Ib.

Set II

For problem IIa, five grade 6 pupils calculated the area correctly but none gave the correct symbolism. Five grade 7 pupils calculated correctly, but only one gave the correct symbolism.

For problem IIb, performance was quite similar to that for problem Ib.

Set III

For problem IIIa, all pupils computed correctly, but only three of the grade 7 pupils gave the correct symbolism.

For problems IIIb, and IIIc, similar performance was noted.

For problem IIId, four pupils at each grade computed correctly, but only two of the grade 7 pupils gave correct symbolism.

4. Summary

"Let us draw some brief conclusions:

a. The auxiliary school pupils are weak in their solution of problems in computing the areas of geometric figures from a drawing. More difficulties occur in determining the dimensions of figures whose length is not expressed by a whole number of the basic units of measure. In these cases, the pupils usually round off the dimensions to whole numbers.

b. The pupils encounter fewer difficulties in determining the areas of geometric figures when the dimensions are given.

c. In more cases, the pupils of the public school give an accurate and precise notation and understand the solution of the problem well, although with weak pupils, some mistakes are encountered." (pp. 43-44)

APPENDIX C: CUBIC MEASURE

Tishin, P. G. /Instructing auxiliary school pupils in visual geometry./ Proceedings of the Academy of Pedagogical Sciences of the RSFSR, 1952, 41, 81-164. (Soviet Studies in the Psychology of Learning and Teaching Mathematics, 1975, 10, 1-124).

1. Focus

To determine the volume of geometric bodies.

2. Procedure

Subjects were six pupils at each of grades 6 and 7 of auxiliary schools and grade 4 of public schools. There were two sets of problems.

"We asked the children to compute the volumes of the following bodies /set I/:

a. a parallelepiped 6 cm long, 4 cm wide, 10 cm high;
b. a parallelepiped 4 cm long, 4 cm wide, 10 cm high;
c. a parallelepiped 2 cm long, 2 cm wide, 20 cm high;
d. a cube whose edge was 10 cm long;
e. a cube whose edge was 2 cm long.

We asked the pupils to solve the following problems /set II/:

a. What is the volume of a rectangular parellelepiped 5 cm long, 3 cm wide and 8 cm high?
b. What is the volume of a cube whose edge is 7 cm long?
c. What is the volume of a classroom if it is 10 m long, 7 m wide and 5 m high?
d. There are eight patients in a hospital ward. How many cubic meters of air belong to each patient if the room is 8 m long, 6 m wide and 5 m high?" (pp. 44-45)

3. Results

Word problems were easier for grade7 pupils than "practical" or measuring problems. The symbolism was usually incorrect. Computing the volume of a cube was more difficult than computing the volume of a parallelepiped.

4. Summary

"Let us draw some brief conclusions:

a. In computing the ... volume of a cube and a parallelepiped of given geometric forms, the pupils encounter significantly more difficulties than in determining the ... volumes of these same forms when their dimensions are given. One can explain this by the fact that in the first case, the pupils approach the solution more creatively, and in the second case, more mechanically, lacking a sufficient number of exercises ...

b. The second inadequacy in the pupils' studies is the absence of an accurate and correct form of notation in solving problems. This indicates that one must pay serious attention to correct notation in solving problems in computing ... volumes.

c. Some pupils do not distinguish such geometric concepts as area, volume, and perimeter. These pupils often confuse one concept with another. One must give this problem serious attention as well.

d. Correct notations is perfectly accessible to the pupils of the auxiliary school, but one must give this problem the most serious attention." (pp. 47-48)

APPENDIX D: SPATIAL FEATURES (part One)

Khanutina, T. N. /Some features of elementary arithmetic instruction for auxiliary school pupils./ Proceedings of the Academy of Pedagogical Sciences of the RSFSR, 1957, 88, 159-185. (Soviet Studies in the Psychology of Learning and Teaching Mathematics, 1975, 10, 183-224.)

1. Focus

To ascertain the knowledge of spatial features of objects (e.g., big/small).

2. Procedures

There were 83 subjects, distributed as follows:

age(years)	7	8	9	10	11
number	15	5	22	30	11

The seven-year-olds were normal pupils; all others were retarded pupils.

"We examined and constructed the pedagogical experiment as a teaching experiment, in which the experimenter's oral instructions guided and directed the child's action with visual aids. The quality of the child's performance with assignments depended on the quality of the verbal instruction; hence, we strove for precision and clarity of instruction.

Moreover, we carefully registered and strictly considered the child's speech accompanying some of his actions or the teacher's actions. We considered the influence of the child's speech on the character of his fulfillment of the assignment, and his correlation of words with actions. When they did not correlate (due to inaccuracy of speech or incorrectness of action), we pointed out the mistake to the child and corrected his speech and actions.

Verbal instructions and help from the teacher directed the pupil's apperception and thinking to the primary, essential aspects of objects and helped divert them from non-essential aspects." (p. 185)

3. Results

In the discussion that follows, P denotes procedures and R denotes results.

a. P: investigate big/small, thick/thin, wide/narrow, long/short, high/low, using sets, each of ten objects, by asking for biggest of the ten objects, then biggest of the nine remaining, and so on.

R: Normal pupils did all the tasks; retarded pupils completed only 60% of the tasks. Six retarded pupils did all the tasks, six did none. For retarded pupils, finding the largest block among blocks sized 4, 3, 2 and 1 cm was easier than for blocks sized 7, 8, 9 and 10 cm. Normal pupils could spontaneously correct errors; retarded pupils did not.

b. P: investigate bigger/smaller, etc. by using two sets of objects; one object in set I identified by experimenter; pupils asked to select all objects from set II which are bigger (smaller) than identified object.
R: Normal pupils did all tasks. Three retarded pupils completed all the tasks; thirteen did none. When pupils were asked to select the object in set II which was the same as the identified object, one retarded pupil did all tasks, and fifteen pupils did none.

c. P: investigate bigger/smaller, etc., when two sets were at different ends of room
R: Normal pupils completed 96% of the tasks. None of the retarded pupils did all of the tasks; twenty-four did none.

4. Summary

Twelve of the fifteen normal pupils arranged the objects by size before performing the tasks. The retarded pupils did not do this without specific instruction for each set of materials; that is, there was no transfer.

APPENDIX F: MATHEMATICAL ABILITIES

Krutetskii, V. A. /The Psychology of Mathematical Abilities in School Children./ (J. Kilpatrick, and I. Wirszup, Eds., and J. Teller, Trans.) Chicago: University of Chicago Press, 1976. (Originally published, 1968.)

1. Purpose

The basic goal was "to create psychological foundations for an active pedagogy of abilities." The specific goals were (a) "To characterize the mental activity of mathematically gifted pupils as they solve various mathematical problems," (b) "to create experimental methods of investigating mathematical giftedness that might have an independent value," (c) to reveal "typological differences in the structure of abilities," and (d) to determine whether the development of mathematical abilities is related to age.

Several specific hypotheses regarding mathematical abilities were also made.

(1) Pupils with different mathematical abilities are characterized by differences in degree of development of both the ability to generalize mathematical materials and the ability to remember generalizations.

(2) Able and less-able pupils differ in their rate of "curtailment" of reasoning.

(3) Pupils with different mathematical abilities are characterized by different degrees of the ability to switch from a direct to a reverse train of thought.

During the course of the study, additional hypotheses were generated.

(4) Able pupils are characterized by an ability to switch rapidly from one mental operation to another.

(5) An ability for spatial concepts is expressed in different ways and may be related to the presence of different types of mathematical abilities.

2. Rationale

The foundation for the study is built very carefully through an extensive review of non-Soviet as well as Soviet research literature. Ability is not viewed as an inborn trait. Rather, certain typological properties are considered to be inborn. The manifestations of these mental properties, however, are determined by the circumstances in which an individual is reared.

Seven basic assumptions are stated:

(1) Abilities are abilities for a definite kind of activity, exist only within a specific activity, and must be studied within that activity.

(2) Ability is dynamic and develops only in a specific activity.

(3) There are optimal age periods for the development of abilities.

(4) Progress in an activity depends on a complex of abilities.

(5) High achievement in an activity can be conditioned by different combinations of abilities.

(6) Relative weakness in one ability can be compensated by other abilities.

(7) General and specific giftedness are related, though the nature of the relationship is not well understood.

3. Research Design and Procedure

The research was conducted over the twelve-year period 1955-67. In all, 201 subjects were studied, some briefly and some for several years. In addition, groups of 62 and 56 mathematics teachers and 21 mathematicians were surveyed, biographies of 84 prominent mathematicians and physicists were studied and data which permitted correlation of progress in various school subjects were examined for more than 1000 students in grades 7 to

10 in Moscow schools. Further data were gathered from several local mathematics contests and from examination of notebooks of a "large number" of students in grades 6 to 8.

Seven basic principles underlay the methods of the study. First, activities were chiefly mathematical in order to highlight mathematical abilities. Second, experimental problems were designed to reflect various degrees of difficulty. Each problem type was presented by a series of problems of increasing complexity and difficulty. The simplest problems were designed to be accessible even to pupils of indifferent abilities. Third, solving the problems should help clarify the structure of mathematical abilities. That is, features of mental activity specific to mathematical activity should be manifested. Fourth, processes for solving problems were more important than the fact of a final solution. Fifth, to measure ability rather than past habits, experience and skills, the problems that were selected were non-standard and required little particular previously learned information. Sixth, experimental methods were used that were instructive as well as diagnostic. The pupil's rate of progress was observed in two situations: (1) independently, and (2) with slight help from the experimenter. Seventh, quantitative as well as qualitative methods were used. Counting data (e.g., number of problems solved with and without help, number of different solutions) were maintained, and factor analysis was applied to aid in the interpretation of the data.

Twenty-six series of problems were developed. Briefly described, they are as follows: unstated question, incomplete information, surplus information, isolation of parts of a figure, inductive generalization, common mathematical structure, increasing abstraction, generalization from a single instance, proof, composition of equations, unrealistic situations, artificial concepts, multiple solutions, changing content, reconstructing a process, unconscious restrictions, direct and reverse processes, heuristics, logical reasoning, series, sophisms, complex data, visual, verbal and visual, spatial concepts, visual-pictorial versus verbal-logical.

Preliminary experimentation was conducted with selected students. Labeling of subjects as very capable (VC), capable (C), average (A) or incapable (I) was accomplished in accordance with broadly stated guidelines. Experiments of the longest duration were conducted with the capable and very capable students. Six major studies involving individual experimentation were conducted.

Subjects were told that the purpose of the experiment was to collect material for new problem books. Usually experiments were conducted individually, during out-of-class time, and after a good rest. Indisposition, fatigue, low spirits, or lack of interest in solving the problems were sufficient cause for postponement of a session.

Years	Grades (Age)	Ability Level			
		VC	C	A	I
1956-58	6-8		9	6	4
1958-65	(6-14)	16			
1960-61	6-7			9	10
1960-64	5-9		36	22	8
1962-64	9-10	11	8	11	
1963-65	2-4	7	14	9	12
1956-65	2-10	34	67	57	34
				TOTAL	192

4. Findings and Interpretations

Several components of mathematical thought were identified by at least 50% of each of the two groups of mathematics teachers surveyed: logical thinking, resourcefulness in studying mathematics, stable mathematical memory, and ability to generalize. Ability to generalize and abstracting essential features of a problem were cited most frequently by the mathematicians surveyed.

On the basis of selected cases of mathematical giftedness, several conclusions were made. First, mathematical abilities can take shape early, usually in the form of computational skills. Second, characteristics that develop include ability to generalize, flexibility in processes, striving for economical solutions, memory for generalizations, curtailment of reasoning, and mathematical perception of the environment. Third, giftedness at an early age is relatively independent of support of such development. /"Flexibility in processes" means an ability to switch rapidly from one operation to another or from one train of thought to another. "Economical solutions" are the easiest, clearest or most direct. "Curtailment of reasoning" refers to the shortening of processes of solution when the processes are used more than once. The logical jumps between explicit steps in these processes become larger as the intermediate steps are accepted as 'obvious.'/

From the data of all subjects (but with primary emphasis given to the data of capable and very capable students) the following conclusions were reached:

(1) Capable pupils perceive the mathematical material of a problem analytically (different elements are assessed differently) and synthetically (relationships are sought for among the elements). Average and incapable students perceive only disconnected facts, and have difficulty synthesizing concrete data. There seems to be an ability to extract from the given terms of a problem the information maximally useful for its solution.

(2) Capable pupils generalize quickly and broadly. They generalize not only the content of the problem, but also the method of solution.

(3) Capable pupils can with very limited exposure to similar problems curtail their processes for solving such problems. Average pupils do this only after repeated exposure. Incapable pupils experience great difficulty in producing such curtailment.

(4) Capable pupils easily switch from one mental process to another qualitatively different one, approach problem-solving from different aspects, are free from conventional solution techniques, and easily reconstruct established thought patterns. Incapable pupils are marked by inertness, sluggishness, and constraint in their thinking and they are impeded by previous solution techniques.

(5) Capable pupils strive for the clearest, simplest, shortest and most elegant solution to a problem.

(6) Capable pupils can reverse their reasoning processes easily.

(7) When able pupils solve a hard problem, their trials seem to be a means of thoroughly investigating it rather than direct attempts at solving it.

(8) Capable pupils remember generalized and curtailed structures. These structures are created from the data and the relationships of particular problems.

(9) There appears to be an identifiable "mathematical cast of mind" which is formed "as a particular synthetic expression of mathematical giftedness and includes cognitive, emotional and volitional aspects." Further, there appear to be three types of "mathematical minds:" analytic, geometric, and harmonic (combination).

(10) Sudden inspiration among capable pupils is frequently explainable by the ability to generalize and the ability to think in curtailed structures.

(11) Capable pupils tire much less during mathematics lessons than during other kinds of lessons.

(12) The ability to generalize appears to develop first. Curtailing reasoning, generalizing memory and striving for elegance in solutions appear to be formed later.

(13) There is no difference in qualitative characteristics of mathematical thinking of boys and girls.

Summary conclusions were as follows:

(1) There appears to be a basis for speaking of specific abilities (including mathematical abilities) rather than general abilities that are only "refracted in a unique way in mathematical activities."

(2) In some people, the brain is uniquely attuned toward isolating from the environment stimuli of the type of spatial and numerical relationships and symbols and toward optimal work with precisely this kind of stimuli. That is, some people have inborn characteristics in the structure of their brains which are extremely favorable to the development of mathematical abilities.

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Chapter 13

UNDERSTANDING OF FRAMES OF REFERENCE BY STUDENTS FROM AGE 11 TO COLLEGE AGE

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The ability of students to properly use frames of references appears to be an important consideration in the development of understandings in euclidean geometry. The research of Piaget, et al., suggest that all students by about the age of 12, should be able to correctly use an external frame of reference to properly predict water level, pendulum position, etc. (Piaget & Inhelder, 1956). In addition, Piaget seems to suggest that the ability to use these systems of reference develop at the same time as a student's ability to apply a coordinate system to a spatial task. This research examines the degree to which students at various age levels have developed these understandings.

Data was gathered on the level of understanding of frames of references of different age level students to verify or point out disagreements with Piagetian findings. Also, evidence was gathered on the ability of students to use a coordinate system.

The study was conducted in 1977. The set of subjects consisted of 375 students from the fifth grade, eighth grade, eleventh grade and university pre-service teachers (PST). The ages of the students ranged from 11 to 12 years of age at the fifth grade level to 20 to 22 years of age at the university level.

The fifth grade level was chosen as the starting point since these children were at the approximate age for which Piaget has suggested an almost universal capability to perform the required tasks.

The subjects were given three paper and pencil reference system tasks. These tasks were administered to groups of students. The tasks were the "bottle" task, the "pendulum" task and the "faucet" task. For each task, the subject was asked to draw the water line or the pendulum position if the container would be tipped as shown in Figure 1.

The second part of the data gathering consisted of short individual interviews with a subset of subjects chosen at random from each group. The purpose of the first part of the interview was to verify the results of the group administered tasks. The total number interviewed approximated 25% of the subjects in the study. The interview session was divided into two parts. First, each subject was asked to repeat the "bottle" task. The subject was then given an opportunity to explain his placement of the water line. To further verify the subject's response, a series of questions and tasks were administered culminating with an actual bottle for those who had responded incorrectly or inconsistently. The responses were then classified according to their consistency and to the extent to which a rule was used.

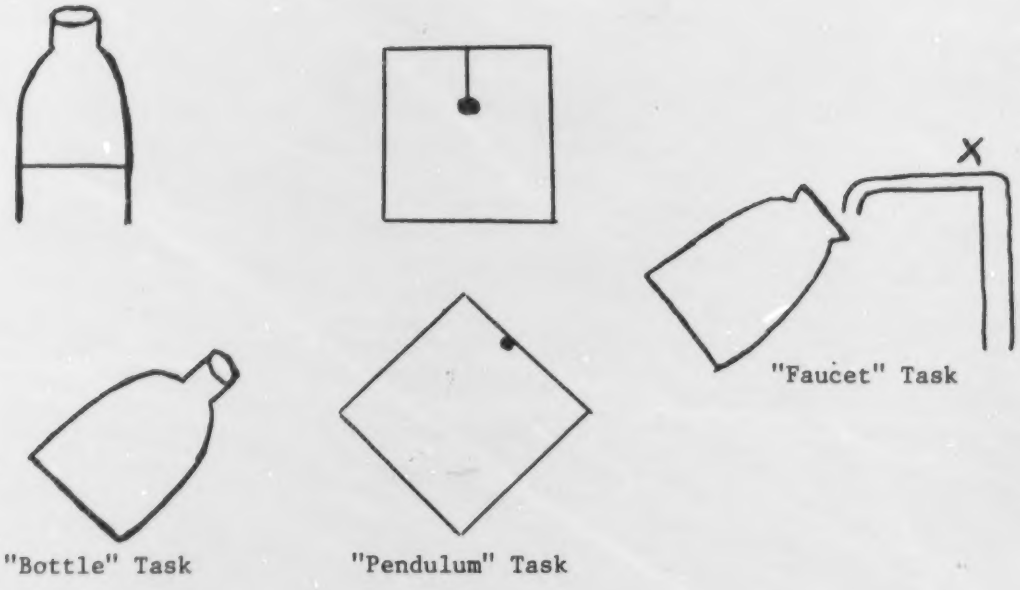

Figure 1. Stimulus for Three Tasks

The second phase of the interview consisted of a point location problem. The purpose of this task was to obtain data on the relationship of the development of the use of frames of reference and the ability to use coordinate systems. Piaget and Inhelder (1956) have indicated that these two abilities develop simultaneously. The task required the subject to look at a piece of plain paper on which four dots had been randomly placed. The subject was asked to locate the four on another piece of paper of the same size. In other words, if one paper would be placed over the other, the location of the dots would exactly coincide. The interviewer noted if the subject used a reference system or approximation in locating the dots. A reference system was judged to be used if measurements were obviously made. These measurements could be rectangular; normally that would mean a measurement from the top of the paper down to the point in question and also a measurement from the side of the paper. The measurement could also incorporate an angular measure or the use of polar-like coordinates. If the placement technique was not obvious, the interviewer would ask for an explanation of tec hnique used. A ruler and protractor were available, but the subjects were not directed to them initially. Their suggested use was mentioned later in the interview.

RESULTS

Group Results. The results of the group administered tasks were surprisingly low, particularly with regard to the bottle and faucet tasks.

The percent figure in Table 2 shows that less than 40% of the 11th graders and the PST group were able to correctly predict water level. For these subjects, at an age which suggests attainment of adulthood, age does not bring with it a capability to use frames of reference with regard to horizontals.

The "pendulum" task was correctly performed by a much larger portion of each of the groups. However, about 30% of 11th graders and the PST group still answered incorrectly. Although it was not the purpose of this study to differentiate between the two task types, some evidence does seem to exist to suggest that horizontal and vertical capabilities are different. A research study designed to focus on this aspect would be necessary to more firmly establish this result.

Table 1 shows an increase in the percent of subjects for all these tasks from the 5th to the 11th groups. This suggests that attainment of these capabilities is developmental. It also should be pointed out that for every task, the increase in percent of subjects giving correct answers stops with the 11th grade group; there is an insignificant decrease from 11th to PST. This fact suggests a leveling off in the developmental growth. Other research is needed to provide further information in this area.

Table 1

Measure of Correct Responses

		TASKS		
Group	n	Bottle	Pendulum	Faucet
5th Grade	107	14.95%	44.34%	9.35%
8th Grade	85	18.82%	50.59%	16.47%
11th Grade	110	32.11%	70.00%	38.18%
PST	73	31.51%	69.86%	31.51%

A measure of the significance of the group difference was obtained by using a Chi-Square test (Table 2). Although a positive growth is apparent, none of the task results showed a significant change from the 5th to 8th group. Whereas, in every instance, the growth from 5th to 11th and from 8th to 11th was significant. A growth period during the early secondary school years after a period of insignificant growth is an interesting result.

A significant increase in use of reference frames occurred from 5th to PST and also from 8th to PST with one exception; the change reported on the bottle task for the 8th grade group to the PST group was insignificant. Caution must be raised at this point. Some care was taken to obtain a representative sample of students for each of the 5th, 8th and 11th grade group. The PST group, however, is a special subset of the college level

population and is not necessarily representative since all subjects in the PST group were elementary education majors. Future research is being planned to study a more representative group of adults.

Table 2

Chi-Square Group by Group Comparison

	G R O U P		
	8th	11th	PST
Group		Bottle Task	
5th	.51	8.80**	6.99**
8th		4.36*	3.40
11th			.01
		Pendulum Task	
5th	.74	14.53**	11.36**
8th		7.64*	6.06*
11th			.00
		Faucet Task	
5th	2.20	24.75**	14.24**
8th		11.04**	4.96*
			.84

* $p < .05$
** $p < .01$

As a part of analyzing the data into correct and incorrect responses, an attempt was made to more carefully classify the data according to a Piagetian classification scheme (Table 3). It is interesting to note changes in error patterns of the different age groups. In the general data anlysis, the two categories "Level" and "Corner Level" were considered together as the correct responses.

The dramatically incorrect "Side" and "Bottom" categories, in particular, become negligible in older students. In a Piagetian scheme these respondents might be considered to be in the Pre-Operational stage.

A rather consistent appearance of a large group in "Up," "Down," and "Corner Down" would suggest that many respondents are possibly just developing into concrete operational thinkers.

Table 3

Response Categories by Nine Categories

Response Sub-categories	Water Line	Group Percents 5th	8th	11th	PST
Level		14	13	21	25
Corner Level		1	6	12	7
Up		20	17	10	18
Corner Up		0	1	0	3
Down		29	27	26	27
Corner Down		20	23	28	18
Side		3	4	0	0
Bottom		13	7	1	0
Vertical		0	2	2	3

Interview Results. The interview was designed to give indepth information with regard to the subjects reaction to the "bottle" task. The comments and reaction of students were recorded and analyzed for rule use. That is, if the subject made a comment such as "the water should be level" or "it will be parallel to the table" the investigators concluded that a correct rule was present. The results were not conclusive, since only one subject from each of the 5th and 8th grade groups indicated a rule, while 12 of the 11th grade group and 8 of the PST group used a rule.

The main objective of conducting the follow-up interview was to verify the results of the group administered tasks. The information obtained in the interview regarding the "bottle" task was compared to the results of

the "bottle" task. If the results were in agreement, either both correct or both incorrect, the results were judged to be consistent. Table 4 summarizes the results as percents. The group results were judged to be a reasonably accurate reflection of the actual response, since 75% or more of each group showed consistency from the group administration to the interview.

Table 4

Consistency of Group Results
with Interview Results

Group	Percent in Agreement
5th	79%
8th	83%
11th	75%
PST	95%

Next, those interviewed were given the point location task as described previously. The procedure of each student was observed to ascertain the use of some frames of reference. The results point to a considerable difference in the capabilities of these students between the "bottle" task and this "dot" task (see Table 5). All but one of the students interviewed from the

Table 5

Use of Reference System in Dot Task

Group	n	Percent Using Reference System
5th	26	61.5%
8th	23	60.9%
11th	28	96.4%
PST	19	100%

11th and PST groups successfully used a reference system for the "dot" task. Whereas, less than 35% of each of these groups successfully performed the "bottle" task (see Table 1). A similar difference in results may be observed for the 5th and 8th groups. These results support the contention that students do not develop the use of these two abilities at the same time. The point location task appeared to be a much easier problem to solve than the horizontal task. Piaget's suggestion that the development of the tasks is sequential; that is, from reference systems to coordinate systems, is not supported by this data. In fact, the sequence may be in the opposite direction!

DISCUSSION

The results of this study disagree with the Piagetian research in terms of when capabilities are acquired. Not only did a majority of 5th graders fail to correctly complete the tasks, but none of the age levels showed a majority with an ability to perform the task dealing with horizontals. These results would seem to suggest that a large number of subjects may never be able to correctly use a frame of reference in a similar situation. Further research is needed to more carefully identify the point of acquisition of such a capability. These results may provide valuable information to help determine teaching strategies to the extent that this information measures a student's ability to think concretely or abstractly.

A second area of disagreement lies in the ability of students to apply a rectangular coordinate system as compared with the horizontal and vertical frame of reference. These results suggest that these capabilities are developed differentially and do not develop simultaneously or sequentially as Piaget's results suggest.

It would appear that the implications are potentially far-reaching. Many research problems are possible and several have been mentioned. The following questions are raised to suggest additional areas.

1. How does the inability of students to use systems of reference relate to environmental models? That is, do horizontal ceiling lines, horizons, etc. communicate a "horizontal" concept?

2. Is there a greater, earlier exposure to rectangular coordinate systems that could introduce a "training" effect to upset a predicted developmental sequence?

3. Does a possible inability to use frames of reference suggest curricular approaches of more concrete nature?

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PART IV

PRESCRIPTION INSTRUCTION

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Chapter 14

TWO LEVELS OF PUPIL VERBALIZATION AND PRESCRIPTIVE INSTRUCTION USING PHYSICAL MODELS

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In recent years, there has been a strong movement toward the use of physical models in teaching mathematics to young children. The work of Jean Piaget (1970), Jerome Bruner (1960) and Zolton Dienes (1967) has contributed to the theoretical framework for this movement. Recent research findings have not, however, shown manipulative approaches to teaching mathematics to be conclusively superior to traditional, purely symbolic approaches. The reasons for this may be that various factors involved in learning through physical models have not been considered. Some of these factors are (1) sequencing of instruction, (2) the appropriateness and relative abstractness of various physical models for symbolic concepts, and (3) the role of language in learning by young children. The present study was an attempt to investigate the third of these factors while holding the other two constant.

Language is an important medium for learning, especially for older children and adults. Its importance for young children, however, is not clear. Infants learn tactually and visually before they understand any language at all. Piaget (1970) has shown that preschool children use language egocentrically rather than to direct and organize their activities. He and his colleagues believe that the child's use of language is determined by his level of cognitive development. According to Inhelder, Bovet, Sinclair, and Smock (1966):

> Language learning does not provide, in our opinion, a ready-made 'lattice' or lens which organizes the child's perceptual world. Rather the lattice is constructed in the process of the development of intelligence, i.e., through the actions of the child on the environment and the interiorization of these actions to form operational structures. (p. 163)

Vygotsky (1962) and many other Russian psychologists (e.g., Luria and Yudovick, 1968) take issue with Piaget. They believe that learning to speak provides a child with a means of organizing experience and expanding mental abilities.

> The child's intellectual growth is contingent on his mastering the social means of thought, that is, language.... All the higher psychic functions are mediated processes, and signs are the basic means used to master

> and direct them. The mediating sign is incorporated in their structure as an indispensible, indeed the central, part of the total process. In concept formation, that sign is the word, which at first plays the role of means in forming a concept and later becomes its symbol. (Vygotsky, pp. 51, 60)

Research involving children in or below the primary grades has not solved these controversies. Attempts to bring children to higher levels of cognitive development through language (or other) training methods have met with mixed results. Suydam and Weaver (1975) state that there is "no evidence that training changes the overall developmental level of the child. The positive effect of training is rarely retained unless the child is already in transition from one level to another" (p. 47). On the other hand, Luria and Yudovich (1968) cite an experiment where language training appears to have dramatically affected cognitive development. In the study, a pair of twins was found who had not had normal exposure to adult language models and was thus retarded in their cognitive and social development. One was placed in a regular kindergarten class and the other was given special language training. The twin who was given special training showed remarkable improvement in the imaginativeness and complexity of his play activities whereas the other twin progressed more slowly. Luria and Yudovich use this as one example of how language expands mental abilities of children.

In more recent research, Shulman (1975) compared verbal training with nonverbal training in teaching four mathematically-related concepts to preschool children. She concluded that language was important in helping young children form the concepts. Prindeville (1971) did not find significant differences between a treatment group of first-grade children who received special instruction through a traditional published workbook series and a second treatment group that received training through language and manipulative materials. Others (e.g., Hollingsworth, 1973) have successfully implemented nonverbal mathematics programs, but their work has been confined to older children using abstract mathematical programs.

Few attempts have been made to determine what factors are involved in maximizing the effectiveness of physical models in elementary mathematics. Even fewer attempts have been made to determine the role that language plays in learning mathematical concepts through physical models. None were found which compared various levels of verbalization in learning through physical models.

Setting up a completely nonverbal program of instruction using physical models would not be feasible. Language, at a minimum, serves a directive function in the classroom (the teacher verbally organizes and directs pupils' actions and attention). In addition, verbal labels are given to concepts and actions. "Nonverbal" mathematics programs developed in the past have relied on symbolic representations which in turn rely on the child's previous experience with language. In manipulative mathematics programs, the medium of instruction cannot be symbolic notation since that is the final goal of instruction using physical models.

The question asked in this study, then, is not whether verbal language should be used in instruction. The question is rather, what kind of language should be used in instruction? Should the teacher encourage pupils to verbalize discoveries, to use new vocabulary in the proper context, to discuss and evaluate? Or is it sufficient for pupils to interact with physical materials, hear the teacher's verbalizations and to respond in nonverbal ways (such as raising hands or pointing)?

These questions are most easily studied in a small-group or one-to-one setting such as in diagnostic/prescriptive or tutorial work. It is not as easy to have pupils engage in a high level of verbalization in a large-group taught by one teacher. Thus, it was decided that a group diagnostic/prescriptive setting would be used in the study and that the following questions would be investigated:

1. When children diagnosed as deficient in their understanding of numeration are taught through a series of physical models, will those who engage in a large number of verbalizations relevant to the instruction show significantly greater mastery of place value than those who are discouraged from making such verbalizations?

2. Will the use of a sequence of four physical models of numeration in instruction significantly increase pupils' understanding of numeration as demonstrated by pretreatment--posttreatment comparisons at either or both levels of pupil verbalization?

METHOD

Subjects. The children in the study were from four second-grade classes in a large elementary school in Phoenix, Arizona. According to the classroom teachers, these children had been exposed to multi-digit numeration in the first grade, but (to the best of their knowledge) had not received their instruction through physical models.

All children in the classes were administered the Place Value Knowledge Survey (PVKS) (Wiebe, 1977, pp. 82-89), designed to test mastery of two-digit and three-digit numeration (up to 150) at the concrete, pictorial and symbolic levels. Responses from children in one classroom were used to validate items on the PVKS and as a comparison to scores in the treatment group. Among the children in the other three classrooms, the 40 diagnosed as having the lowest level of mastery of numeration but who successfully answered three of the five readiness items on the instrument were chosen for the study. These children were randomly assigned to four groups, two of which were taught by one teacher and two of which were taught by a second teacher. Because of logistic difficulties, it was impossible to increase the sample size and randomly assign pupils to a control group.

Procedure. The independent variable in the study was the amount of pupil verbalization which occurred during prescriptive instruction. Each

teacher taught one verbal group and one nonverbal group with ten pupils each. In the verbal groups (Treatment 1), pupils were continuously encouraged to answer a variety of questions relevant to the concepts being studied. They were asked to describe, explain and use the vocabulary words introduced by the teacher either individually or as a group. In the nonverbal group (Treatment 2), the subjects responded nonverbally to the teacher by raising hands, pointing, demonstrating; they followed the teacher's model; and they demonstrated various findings to each other. In each treatment, the teacher gave instructions, talked about what the children were doing, used new vocabulary, told stories (relevant to the curriculum) and sought the desired actions from the pupils.

In order to verify that the intended level of verbalization was being carried out in the treatment sessions, the researcher randomly selected one of the two sessions in progress at a given time for observation. During this time, he recorded the number and type of statements made by the teacher and pupils. Sessions not directly observed were tape recorded. The recordings were not analyzed in detail. Instead they were used to give feedback to the teachers of the treatment groups about the level of verbal interaction and their use of the lesson plans during the session.

The treatment sessions were conducted for 16 consecutive school days in the months of September and October, 1976, and lasted about 25 minutes each. The order in which the treatments occurred (e.g., verbal group followed by nonverbal group) and the time of day when the sessions were given were varied so that these factors would not contribute to differences between the treatments.

All four groups received the same instructional content. Four physical models--the bundles-stick model, the bean-stick model (numberation-block), the colored-chip model, and the bundled-stick model (see Wiebe, 1977, pp. 40-42)--were used to progressively develop the concept of multi-digit numeration. There were enough materials so that each pupil could participate in all manipulations. No symbolic representations were introduced until the ninth session, and from that point on, were used only in conjunction with the manipulatives. During the study, the regular classroom teachers agreed not to discuss concepts related to numeration with their pupils.

The dependent variables in the study consisted of posttreatment scores on the PVKS, a concrete criterion test, a symbolic criterion test, a vocabulary test, and a transfer test. The PVKS was administered to groups (as it has been in the pretesting) and the other posttests were administered individually. The order in which the children received the criterion tests was randomized. Scores of the two nonverbal groups were pooled as were scores of the two verbal groups in order to counterbalance teacher effects.

ANALYSIS AND RESULTS

Level of Verbalization. The average number of instructions given to pupils, (e.g., "Susie, point to the hundreds place,"), statements made

(e.g., "That is a bundle of ten."), and questions asked (e.g., "Johnny, how many ones are there here?") by the teachers per observed session are displayed in Table 1.

Table 1

Average Number of Observed Utterances per Session

	Treatment 1 (Verbal)	Treatment 2 (Nonverbal)
Instructions	19.3	39.8
Statements	49.3	60.1
Questions	44.6	15.1
Answer Questions	47.3	1.1
Ask Questions	6.3	2.3
Unsolicited Statements	39.4	7.4

The table also contains the average number of questions answered by pupils, the average number of questions they asked, and the average number of unsolicited statements made (e.g., Jane to Mike, "I have all yellow chips on my abacus!") per observed session. For purposes of this study, a "statement" was defined as a phrase, a one-word response, or a counting sequence. A group response to a question was counted as one statement.

The data indicate that the total number of responses made by the teachers per session in both treatments was very similar, though in Treatment 2 the teachers made more statements (instructions) which called for an action by the student than questions whereas the opposite was true in Treatment 1. The number of verbalizations made by pupils in the two treatments was quite different, as would be expected. Children in Treatment 1 made more verbalizations of all kinds (especially answering questions) than did children in Treatment 2.

Pretreatment--Posttreatment Comparisons. In order to determine if the sequence of instruction used in the study increased students' understanding of numeration, pretreatment scores were compared with posttreatment scores on the PVKS for the treatment groups and for a comparison group. The comparison group consisted of students in the fourth class (which was tested but not included in the study) who scored below the cutoff point used in the other three classes to determine who would receive prescriptive instruction. Comparisons between mean scores on pretests and posttests were made using correlated t tests. The results are shown in Table 2.

Table 2

Pretreatment--Posttreatment Comparisons of
Place Value Knowledge Survey Scores

Value	Nonverbal	Verbal	Comparison
$\bar{X}$	11.95	11.95	10.44
$\bar{Y}$	18.55	16.95	12.20
sd_x	2.40	2.16	2.38
sd_y	2.48	3.49	3.56
r_{xy}	.31	.15	.26
t	-10.27	-5.87	-2.35
n	20	20	25
P-value	.00	.00	.03

At the $\alpha = .01$ level, there was a significant increase in PVKS scores in both treatment groups but not for the comparison group.

Between-Treatment Comparisons. Comparisons between the two treatment groups on the five posttests were made using analysis of variance. The results are shown in Table 3.

Table 3

Analysis of Variance Table
Posttreatment Comparisons

	SS	df	MS	F
1. A (Between Groups)	.02	1	.02	.21
2. Subjects within Groups	3.58	38		
3. AB (Interaction)	.15	4	.04	1.87
4. B x Subjects within Groups	3.01	152	.02	

Neither the F value for between group differences (A) nor the F value for treatment-test (AB) differences was significant at the $\alpha = .05$ level. Based on this result, it was decided not to conduct a posteriori comparisons. Between-test differences (B) were not of importance in this study and, thus, are not reported. The means and standard deviations for the two treatment groups on the four criterion tests are shown in Table 4.

Table 4

Posttreatment Means and Standard Deviations

Value	Concrete	Symbolic	Vocabulary	Transfer
$\bar{Y}_V$[a]	7.70	5.30	6.65	4.75
sd_V[a]	2.30	1.84	1.23	1.77
$\bar{Y}_{NV}$[b]	7.85	6.00	6.75	4.15
sd_{NV}[b]	2.66	1.75	.85	1.31

[a]The subscript "V" represents "Verbal Group"
[b]The subscript "NV" represents "Nonverbal Group"

Even though none of the differences are significant, means for the nonverbal treatment are higher in four out of the five posttests.

CONCLUSIONS

Significant differences between pre- and posttest means on the PVKS in the treatment groups appear to show that the sequence of manipulations used in the study is an effective way of teaching numeration to primary-level children. The fact that pre-post increases in PVKS means were not significant for the comparison group would seem to support this conclusion though design limitations in the study (for reasons mentioned earlier) preclude direct, statistical contrasts between comparison-group, treatment-group results.

The nonsignificant differences between posttest means of children in the two treatments can lead to a variety of interpretations. The most obvious interpretation is that the children learned the numeration-related concepts primarily through the physical models, aided by teacher verbalization, and that their own verbalizations did not help them master the concepts. Indeed, since means of four out of the five posttreatment tests were higher for pupils in Treatment 2, pupil verbalization may have interfered with their mastery of numeration taught through physical models. However, since posttest differences were not significant, the observed differences between the means may have been due to change alone and not to differences in impact of the treatments. Replication with larger sample sizes might answer this question.

Speculations as to why posttest scores in the verbal treatment were not significantly higher than posttest scores in the nonverbal treatment involve time, erroneous verbalization, or teacher effort. In the nonverbal treatment, it took more time for teachers to cover the material because they had to ask for a large number of individual verbal responses, wait for answers, etc. In the nonverbal groups it was easier to obtain

group responses. Thus, there was more time for the children to manipulate the materials.

Studies by Crowcroft (1973) and Schwartz (1954) showed that incorrect verbalization can interfere with discovered generalizations. In the present study, pupils in the verbal groups often appeared to answer questions impulsively. In the nonverbal groups, however, when asked to give a physical demonstration, the pupils appeared to be more reflective. The difference in the number of erroneous verbalizations may have counter-balanced any positive effects verbalization had on learning numeration through the models.

A third factor might involve the experimental effect. Both teachers found it contrary to their natural way of teaching to encourage nonverbal responses from the pupils and to refrain from asking questions that could be answered verbally. It took an extra effort on their part to successfully carry out this method. This extra effort may somehow have affected post-test scores in favor of Treatment 2.

In any case, it is clear that in the present study, a high level of pupil verbalization did not produce a greater mastery of the concept of numeration than did the approach where pupils responded in nonverbal ways. This result seems to contradict efforts of certain educators to encourage "high level" verbal responses from students (Bloom, 1956; Clegg, 1971), at least with children in the "stage of concrete operations."

The above results have implications not only for those working with children individually or in small groups, but also for the classroom teacher in large group settings. Manipulation of physical models and questioning techniques leading to nonverbal responses can be used in large as well as smaller groups. Such questioning focuses each child's attention on the manipulations and concepts involved without disrupting thought processes of other children. In the nonverbal approach, children are not being bombarded as often with the erroneous responses of their classmates.

This study also has implications in diagnostic/prescriptive settings. The results seem to indicate that when using physical models to teach concepts that have already been introduced to (but not mastered by) children, it is probably not necessary to encourage a high level of verbal responses in order to maximize learning. Rather, it may be sufficient for the child to engage in manipulations and for the instructor to supply verbal directions and labels. For children who are shy or who have low verbal ability, a less verbal approach may prove to be more comfortable and thus, may result in more learning than in a highly verbal approach.

The study described in this paper is only a small step toward determining the importance of language when learning through physical models. Further study is needed using children being introduced to mathematical concepts for the first time. Also, various other levels of pupil and teacher verbalization and nonverbalization should be studied. Finally, there is a need to look deeper into all phases of the use of physical models in teaching mathematics. Which sequences are best for which children? How long should use of physical models continue before children are required to work only with symbols?

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Chapter 15

THE EFFECTS OF A SUPPLEMENTAL CLINIC ON CLASSROOM MATHEMATICS ACHIEVEMENT ATTITUDE AND SELF-CONCEPT

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The supplemental diagnostic/remedial clinic is a relatively new idea in the area of mathematics. It is generally recognized in the education community that numerous children have learning deficiencies in mathematics. It is acknowledged (Wilson, 1961; Bernstein, 1964) that children are not receiving sufficient help in the classroom to overcome these problems, and positive attitudes are not being produced. Teachers have cited "diagnosis of children's learning problems" as one of their major weaknesses in the classroom. Remedial programs that were diagnostic/prescriptive but non-supplemental (New York University Center for Field Research and Services, 1972; Lyon, 1975; Akron Public Schools, 1970) have been investigated as a method of improving achievement. Studies have been made to determine the effects of improved achievement on increased attitude (Abrego, 1966; Blezinger, 1975). It has also been assessed that remedial nonsupplemental programs improve a child's self-concept (Smith & Hess, 1974; Forte, 1975; Fink, 1962; Crawley & Evans, 1974). None of the aforementioned studies, however, examined the effects of a supplemental diagnostic/prescriptive mathematics clinic upon a participating child's achievement, attitude and self-concept.

In the Fall of 1976, such a study was conducted. Twenty-one students 8 to 15 years old who were enrolled in the Clinic in Mathematics Education at Arizona State University (treatment group) were matched by age, grade, and sex to an equal number of students who were on the waiting list for the clinic (control group), to determine changes in mathematics achievement, attitude and self-concept.

In order to measure the three variables, the Key Math Diagnostic Arithmetic Test (Connolly, Machtman & Pritchett, 1971), the Revised Math Attitude Scale (Dutton, 1962) and the Self-Appraisal Scale (Davidson & Greenberg) were each individually administered to both the treatments and the controls a total of three times. The pretest was given at the beginning of the clinic sessions. An initial posttest ($Posttest_1$) was administered at the completion of the sessions and a second posttest ($Posttest_2$) was given six weeks after the termination of the clinical sessions. The scores on these tests are reported in Table 1.

Table 1

Mean Scores on Three Instruments
for Treatments and Controls

Group	Pretest	$Posttest_1$	$Posttest_2$
		Mathematics Achievement	
Treatment	139.00	157.00	153.95
Control	143.90	150.64	152.68
		Attitude Toward Mathematics	
Treatment	-.351	-.031	-.081
Control	.121	.341	.241
		Self-Concept	
Treatment	53.85	57.30	57.30
Control	54.50	56.45	56.00

Analysis of variance was used to determine (a) if differences between the two groups of concerning mathematics achievement, attitude toward mathematics, and self-concept were significant, and (b) if any differences between the two posttests or between the pretest and each of the posttests were significant. The summary statistics for this analysis are in Table 2.

Table 2

Univariate F. Scores for the Three
Dependent Variables

Source	Mean Square	F1,40	p
	Mathematics Achievement		
$\bar{X}$ Pre vs. $\frac{X\ Post_1 + X\ Post_2}{2}$	531.67	14.52	.01*
$\bar{X}\ Post_1$ vs. $\bar{X}\ Post_2$	.30	1.27	.27
	Attitude Toward Mathematics		
$\bar{X}$ Pre vs. $\frac{\bar{X}\ Post_1 + \bar{X}\ Post_2}{2}$	20.72	1.71	.20
$\bar{X}\ Post_1$ vs. $\bar{X}\ Post_2$	136.00	6.33	.02*

Source	Mean Square	F1,40	p
		Self-Concept	
$\bar{X}$ Pre vs. $\frac{\bar{X}\ \text{Post}_1 + \bar{X}\ \text{Post}_2}{2}$	.21	2.01	.16
$\bar{X}\ \text{Post}_1$ vs. $\bar{X}\ \text{Post}_2$	1.08	.13	.72

*Indicates significance

Two significant differences were noted. The scores on mathematics achievement between the two groups were significant between the pretest and posttest$_1$ and the pretest and posttest$_2$. It was concluded, therefore, that children who participate in supplemental diagnostic/prescriptive mathematics activities show more changes in mathematics achievement at the termination of their sessions and six weeks after their conclusion than do children who have not received such instruction. Significance was also found when comparing the two groups with respect to attitude toward mathematics six weeks after the supplemental sessions. There were no significant differences between the two groups with respect to self-concept. In light of these findings, there are many implications for future research.

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Chapter 16

OVERVIEW OF RESEARCH ON LOW STRESS ALGORITHMS

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INTRODUCTION

This presentation is an overview of low stress research and its conceptual background. This includes:

1. A very brief review of the nature and purpose of low-stress algorithms. Some new information on calculating in equation mode is presented, and the preferred symbol for low-stress division is considered.

2. A description of the most recently researched low-stress procedure. This is a procedure for generating and learning basic addition and subtraction facts. In this description, the related research and a modeling system to be introduced by the fact generating techniques are outlined.

3. A description of new low-stress procedures. These are a modification of the low-stress multiplication and division fact generating procedure to allow estimation of the quotient in long division, two levels of low stress short division and low stress factoring division.

4. A description of the structure and results of standard inferential low-stress research and problems of its coordination and availability.

WHAT ARE LOW-STRESS ALGORITHMS AND WHY

Low-stress algorithms are designed to allow easy acquisition of basic numerical competencies and a clear perception of the meaning of basic operations. This means that:

1. Normal children make a large saving of curriculum time.

2. Special, remedial or disadvantaged children can acquire a normal level of mastery in critical numerical competencies.

In general, low stress procedures concisely record every binary operation, so all work is displayed and allow the separate completion of different kinds of component operations, rather than the conventionally required alternation of them. For an example of low-stress addition, see Figure 1. For an example of low-stress subtraction, see Figure 2. For an example of low-stress multiplication see Figure 3. An example of horizontal low-stress multiplication is shown in Figure 4. An example of low-stress division is shown in Figure 5. An example of low-stress division with under- and over-estimates is shown in Figure 6.

$$\begin{array}{r} 6\ 4\ 8 \\ 7\ 6\ 3 \\ 3\ 7\ 8 \\ 9\ 4\ 6 \\ 8\ 7\ 4 \\ +\ 3\ 7\ 6 \\ \hline \end{array}$$

$$\begin{array}{r} \textcircled{3}\ \ \textcircled{3} \\ 6_{9}\ \ 4_{7}\ \ 8 \\ ,7_{6}\ ,6_{3}\ ,3, \\ 3_{9}\ ,7_{0}\ \ 8_{9} \\ 9_{8}\ \ 4_{4}\ ,6_{5} \\ ,8_{6}\ ,7,\ \ 4_{9} \\ +\ \ 3_{9}\ \ 7_{8}\ ,6_{5} \\ \hline 3\ \ 9\ \ 8\ \ 5 \end{array}$$

$$2\ 7\ 8 + 3\ 9\ '7_{5} + 8\ 6\ 4_{9} =$$

$$2\ \textcircled{1}7_{8}\ 8 + 3\ 9\ ^{+}7_{5} + 8\ 6\ 4_{9} =$$

$$\textcircled{2}\ \textcircled{1}\ 2_{4}\ '7_{8}\ 8 + 3_{7}\ ^{+}9_{7}\ ^{+}7_{5} + '8_{5}\ ^{+}6_{3}\ 4_{9} = 1539$$

Figure 1

Vertical and Horizontal Low Stress Addition

```
 21005
-13709
```

```
 21005
 1'099'5
-13709
```

```
 21005
 1'0   '5
-13709
```

```
 21005
 1'099'5
-13709
 ------
  7296
```

21005 - 13709 =

1'0 '5
21005 - 13709 =

1'099'5
21005 - 13709 =

1'099'5
21005 - 13709 = 7296

Figure 2

Vertical and Horizontal Low Stress Subtraction

190

```
  6879
   x97
```

```
  6879
   x97
  ----
  57
   42
```

Note, multiplication proceeds from left to right.

```
  6879
    97
  ----
  5768
   4231
```

Note, conventional or low-stress addition may be used.

```
  6879
   x97
  ----
  5768
   4231
   4546
    2693
  ----
  667263
```

```
  6879
   x97
  ----
  5768
   4231
   4546
    2693
  ----
  667263
```

Figure 3

Low Stress Multiplication

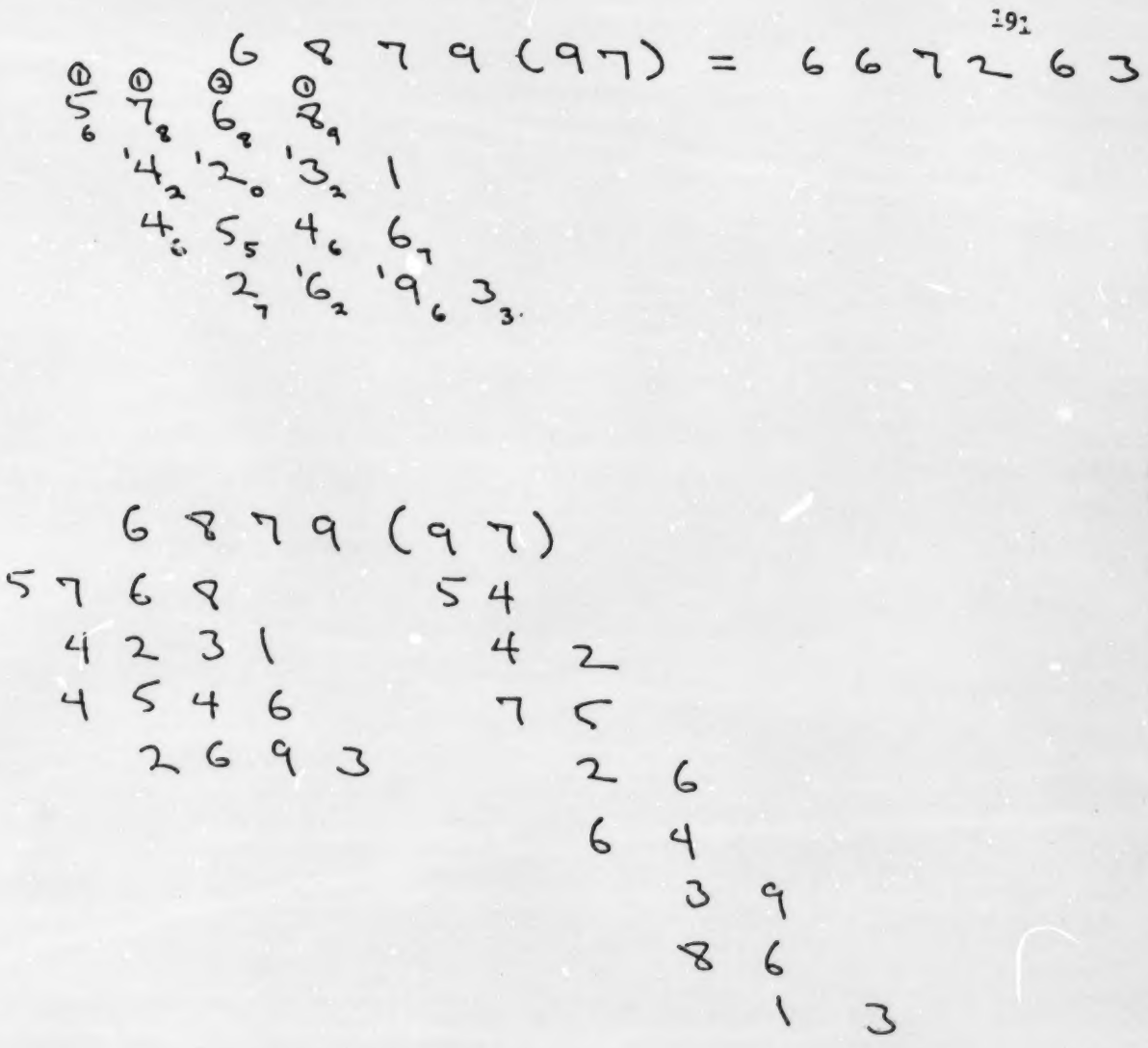
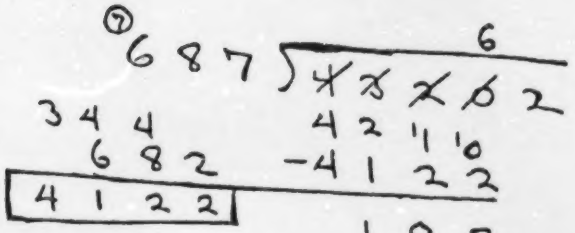


Figure 4

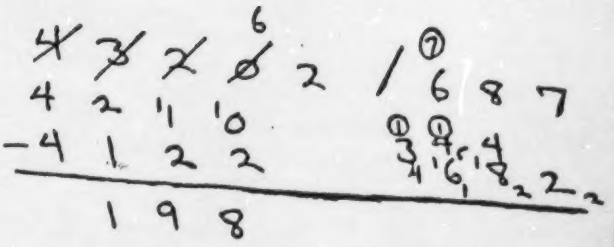
Horizontal Low Stress Multiplication Completed Without Checking and Partially Completed With Checking

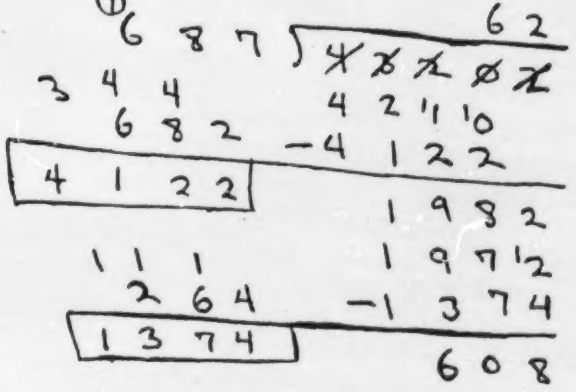
It is not necessary to add both sets of partial products in order to check the multiplication component. Rather, prior to adding any partial products, the identity of respective reversed columns can be easily observed.

Acceptable Form



Recommended Form





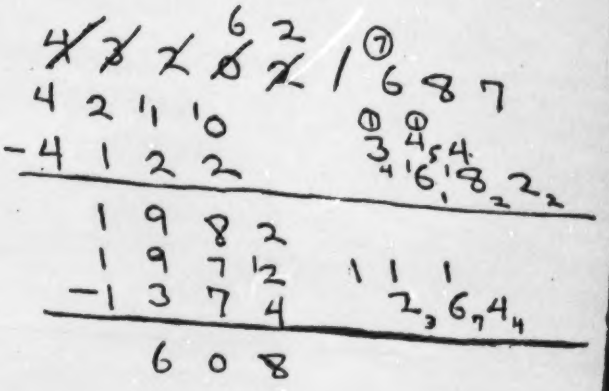


Figure 5

Low Stress Division

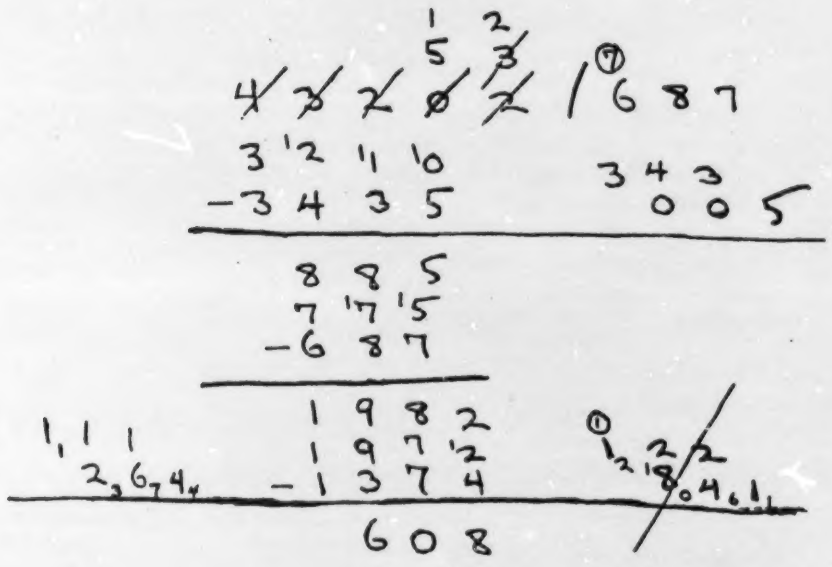


Figure 6

Low Stress Division With Under And Over Estimates

The preferred notation for low-stress division is a/b. The reasons for this are beyond the scope of this paper, but this format has the potential for greatly facilitating the transference from whole number division to fraction procedures and concepts. A trivial but enjoyable advantage of this notation is a mnemonic diagram of operation meanings. (See Figure 7.) If tilting its sign means accelrating an operation and if removing half a sign means reversing an operation and if removing half a sign means reversing its operation, then the system in Figure 7 is clearly derivative. (See Figure 8.) This has the simple elegance of Moorish arithmetic and suggests that the most common division symbol ( ) may have resulted from the drifting calligraphy of medieval rote copy, much as our nonrepresentative arabic numerals did. Something like the following:

1. Assume the original Moorish notation was a\b.
2. This was adjusted by monks to b/ a because placing the usually shorter divisor first made planing the line easier.
3. Then, the notation became b⌐a because materials were expensive and with some separating lines more work could be placed on a page.
4. Then, as 1 2 3 4 5 6 7 8 with quantity representing angles became the pretty but nonrepresentative 1 2 3 4 5 6 7 8 9 so ⌐ became ⌐ and any resemblance to what may have been the original system was lost.

GENERATING AND LEARNING BASIC FACTS IN ADDITION AND SUBTRACTION

Low-stress procedures for generating the addition and subtraction facts should also be a method of learning those facts. There are three versions of the addition-subtraction fact procedure, each appropriate to a different academic level. Each version employs a number line that initially uses discrete counters rather than continuous intervals and a counting sequence designed to eliminate "overlap error."

When constructing the number line, the arabic numerals to 20 are ordinally arranged, with a single counter next to each. The counters could be boxes or any other simple image, however, flower petals are recommended for two reasons:

1. They are easily drawn, even by children with poor motor skills. Sloppy ones are not conspicuously sloppy and there is an intrinsic propriety in making them colorful or interesting.

2. Flowers with five petals translate very smoothly into a later, low stress concept modeling procedure which employs petals and symmetric five-petaled blossoms as an alternate system of numeration.

This system is designed to be easily drawn and easily perceived. The blossom and petal system, called "flower figures," has the representative advantage of the original arabic numerals and modern Cuisinaire rods, plus the biquinary advantage of the Chinese, and the more modern Japanese, abacus. The blossom and petal system is designed to enjoy nonpositional expressions of order of magnitude. This is facilitated by the ease with which petals

Figure 7

Mnemonic Diagram for the Meaning of the Operations

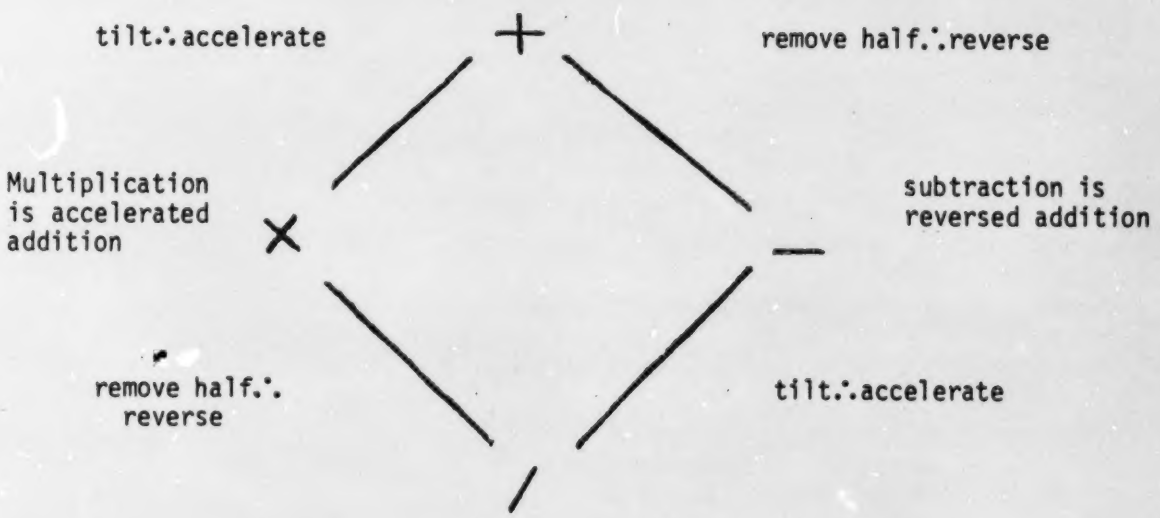
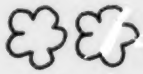


Figure 8

Meaning of the Operations

of adjacent power levels can be made to approximate a 10:1 area ratio which is possible because a single "flower figure" petal closely enclosing a "flower figure" blossom of the next lower power level will approximate the area of two blossoms at that level, see Figure 9, Part A. However, the most important advantage is a one-to-one correspondence between low stress "working" notation and "flower figure" notation, e.g.,

C =  . 10 = '0

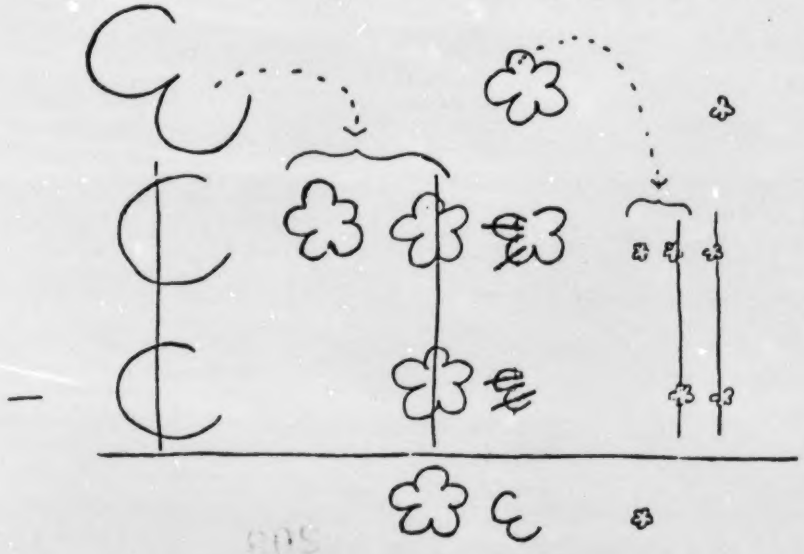
one ten ten ones one ten ten ones

But whether petals are used or not, the number line-counter diagram (Figure 10), in all three variations, is called the Difference Operations Over Format and always referred to, with children, by the acronym DOOF.

Besides the number line of counters, the low-stress addition and subtraction fact generating procedure employs a counting technique designed to eliminate "overlap error." This technique has two forms. For the basic form, students simply place a finger of the hand not holding the pencil over the larger number of a problem and count out the other number on the counter scale. Form A of the diagram (refer to Figure 10) is recommended for use with this procedure, which is demonstrated in Figure 11. For the second form, which is much more mature, a closely related procedure called "zero point counting," is used. In this, the student names the larger number of a problem as the "zero point" of a beginning count and counts out the other number of the problem from that point. This is demonstrated in Figure 12. Forms B_1 and B_2 (refer to Figure 10) of the DOOF are recommended for use with this procedure. When presented together and used alternately they also develop a conceptual base for the first quandrant.

Part A

Regrouping with the Blossoms and Petals of "Flower Figures"



Part B

Regrouping with Arabic Numerals

$$\begin{array}{rrr} 2 & 5 & 4 \\ 1 & {}^{1}\!/_{4} & {}^{1}\!/_{4} \\ -\,1 & 7 & 9 \\ \hline & 7 & 5 \end{array}$$

Figure 9. Correspondence Between Low Stress Working Notation and Flower Notation

QUOTIENT ESTIMATION, LOW STRESS SHORT DIVISION AT TWO LEVELS AND LOW STRESS FACTORING DIVISION

There is a low-stress procedure for generating basic multiplication and division facts. This allows the student to proceed with more advanced problems and calculations while still mastering the basic facts. Moreover, use of the fact generating procedure tends to teach both the facts and the meaning of the operation. The procedure requires simply writing a column of 10 repetitions of the relevant factor. Only nine are necessary for basic facts, of course, but there are further applications of this format with fractions. At that point, having 10 repetitions is an advantage. It should be noted that the usual line, + sign, and column total are omitted to facilitate concentration on relevant partial sums. The low-stress multiplication fact generation procedure is illustrated in Figure 13 and the low stress division fact generation procedure is illustrated in Figure 14. A card with the complete (0 through 9) set of ten, ten times repeated additions is called a Multiplicative Operations Overt Format and always referred to, with children, by the acronym MOOF.

An important application of this fact generating format derives from a small procedural modification which allows the accurate estimation of quotients in long division. When a divisor is rounded to a digit for estimation, the relevant MOOF column is referred to and the student seeks the largest column sums which is less than or equal to the relevant two digits of the dividend, then counts up the factor (addends) from that point. (See Figure 15.) The low-stress quotient estimation procedure strongly assists students not yet facile with division facts or abstract comparison. It also has an interesting potential for short division. It allows the decomposition of a product into appropriate low-stress partial products. (See Figure 16.) Because low-stress partial products are true products, that is because they are factorable, the missing factor of the multiplication can then be easily obtained. This is the quotient, of course. (Note that "partial products" of conventional multiplication are not factorable because they are actually the sums of associated low-stress partial products.)

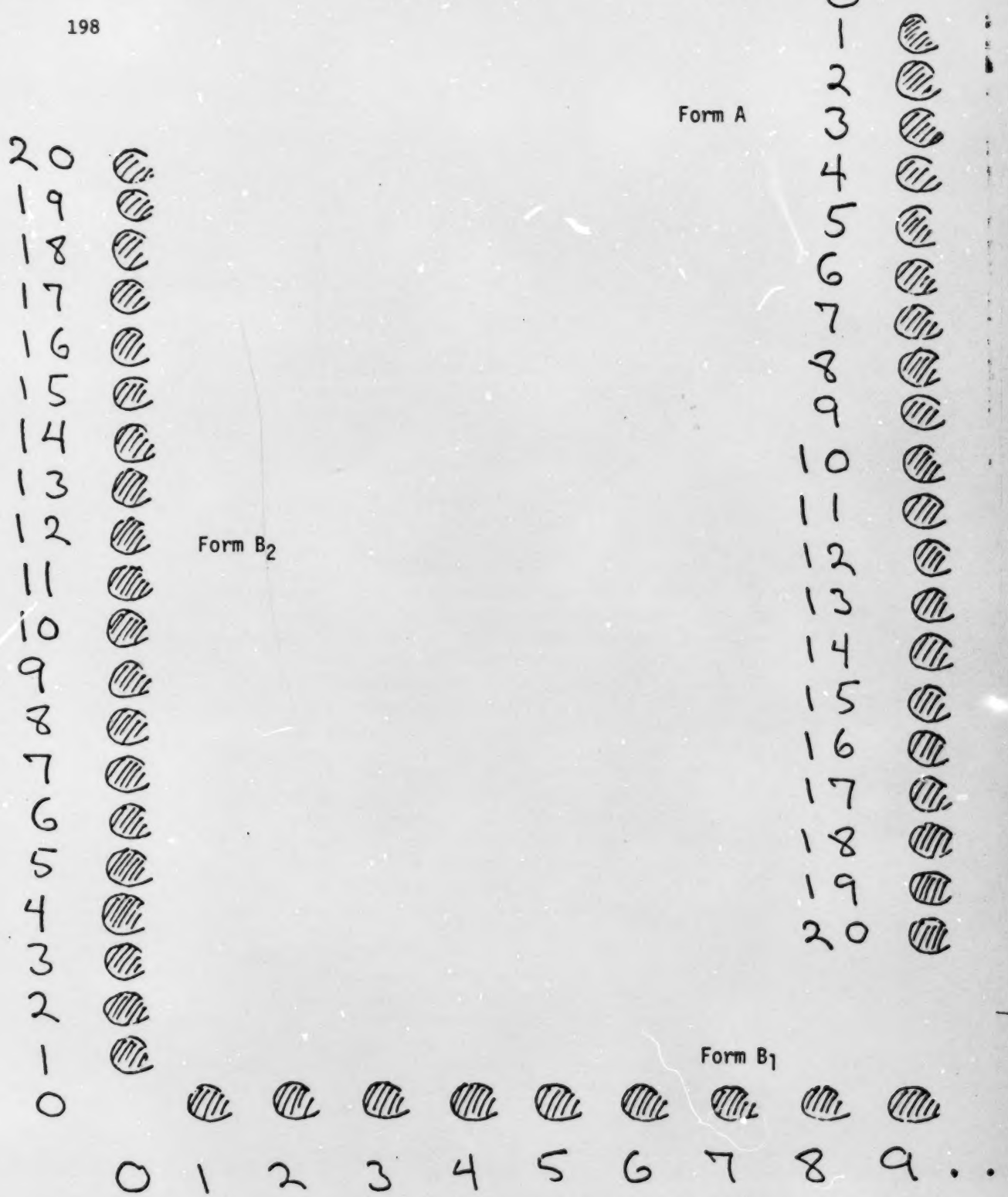


Figure 10

Three Variations of the Number Line Counter Diagrams

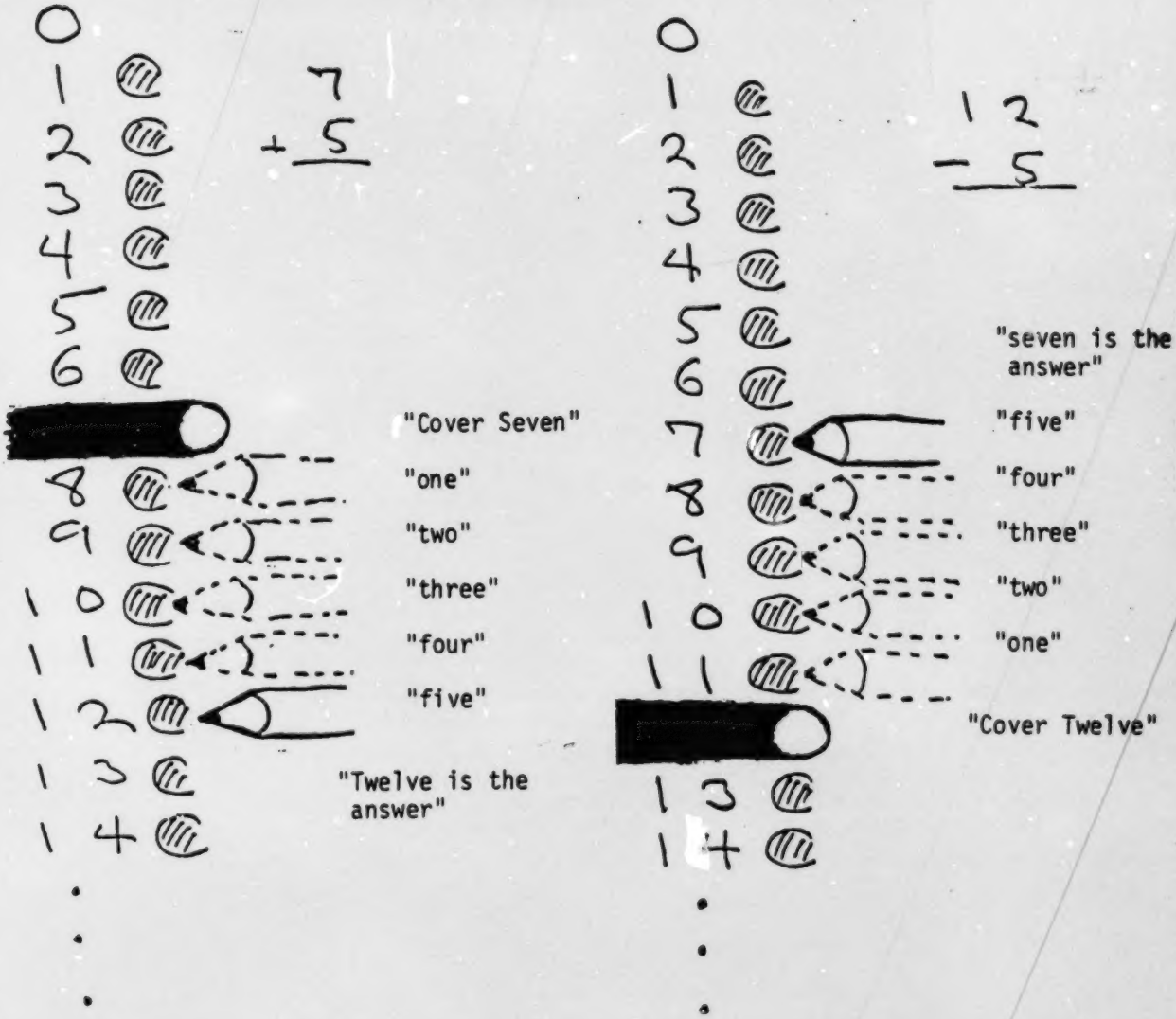


Figure 11

The Generation of Basic Addition and Subtraction Facts Through Simplified Zero Point Counting

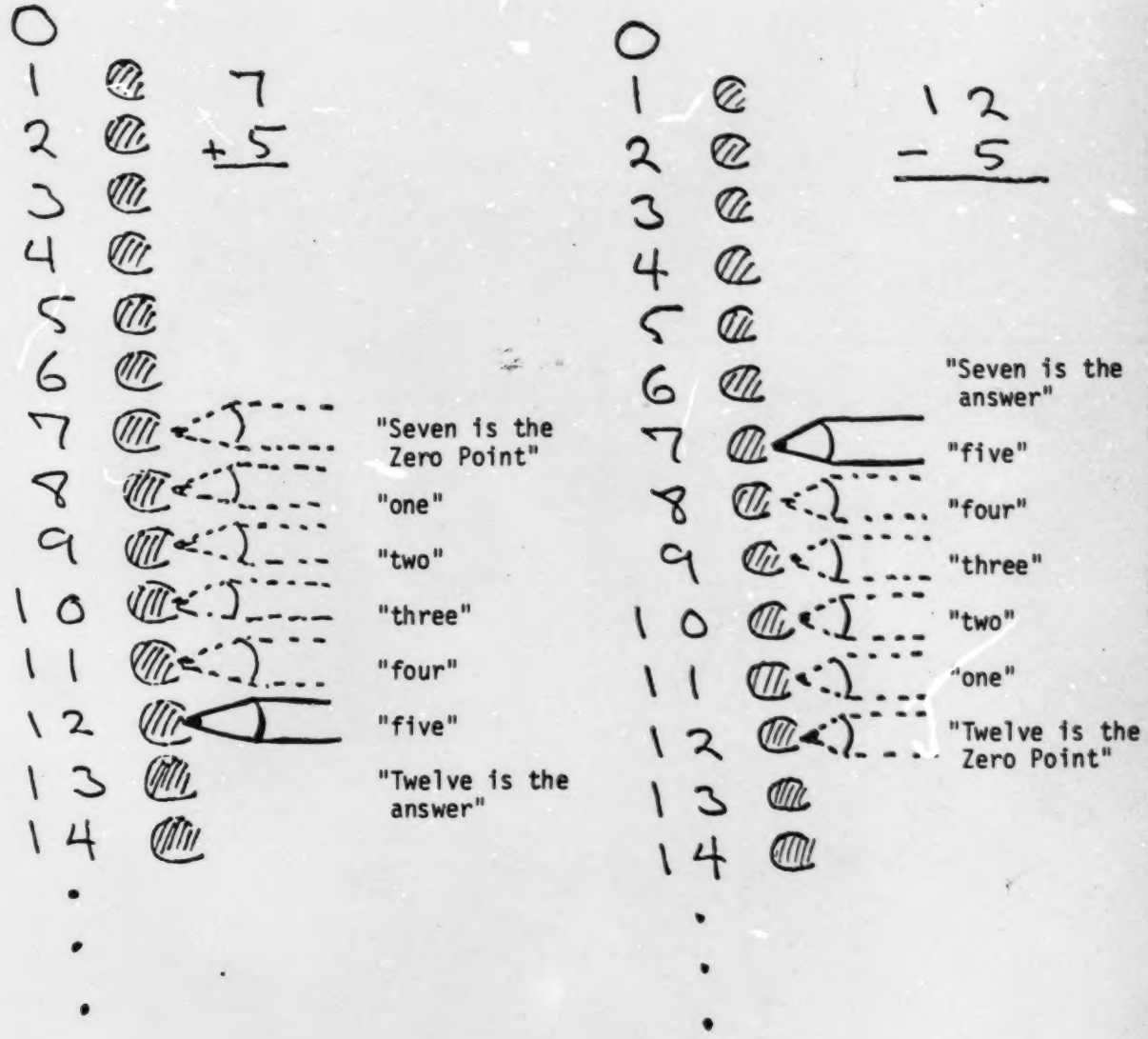


Figure 12

Generation of Basic Addition and Subtraction Facts Through Zero Point Counting

For multiplication one would use repeated additions by counting down the number of addends, finding units, and then counting up the tens

6 x 7 =

We have six times seven. Count down. One, two, three, four, five, six. Okay, four tens and two units is the answer.

6 x 7 = 42

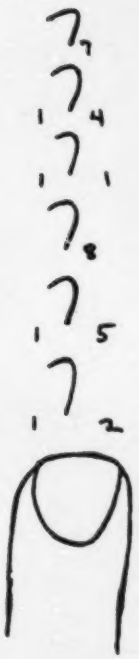
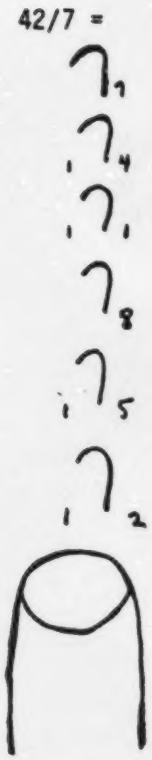


Figure 13

Low Stress Procedures for Learning Multiplication Facts



There are four tens and two ones.

One, two, three, four tens...

Is there a two in units place? Yes.

Count up.

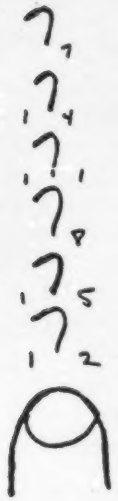
One, two, three, four, five, six.

42/7 = 6

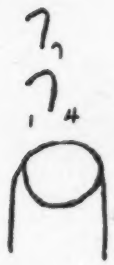
Figure 14

Low Stress Procedure for Learning Division Facts

For the reverse of division the process is reversed by counting down the tens, finding the unit, and then counting up the factors (addends).



"The first two numbers of the dividend say 43. The largest sum less than 43 is 42. One, two, three, four, five, six. Six is the quotient."



"The first two numbers of the remaining dividend say 19. The largest sum less than 19 is 14. One, two. Two is the quotient."

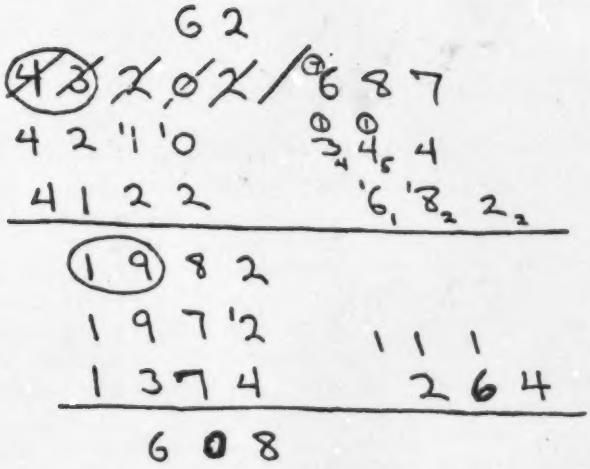


Figure 15

Estimating Quotients for 43202/687

This level of low stress short division is convenient only when there is no regrouping (sums > 10) within the partial product. When such regrouping occur, two convenient modifications are available, one is considered to be the next level of low stress short division and the other is called low stress facoring division. The next level of low stress short division uses half space notation to make the association of relevant subtractive pairs conspicuous. This resembles concisely expressed long division by a digit. Differences involve the notation used, the sequence of steps followed, and the use of low stress quotient estimation. For example, see Figure 17 for 27279/7. Low stress factoring division is procedurally distinct and involves reversing the direction of the regrouping that occurs in level one of low stress short division (refer to Figure 16). The direction of regrouping in low stress factoring division is the same as that in subtraction. For example, see Figure 18 for 27279/7.

Low stress short division never involves more than two places in a regrouping step and will divide both multiples and nonmultiples of the divisor.

Low stress factoring division may involve more than two places in a regrouping step and divides only multiples of the divisor. For nonmultiples it functions as a test of divisibility. Unlike most tests of divisibility, it is procedurally constant for all divisors. Its working notation is briefer than low stress short division, with no loss of record and when trivial regrouping rules apply (e.g., 2, 5) it is extremely fast.

Either method is notationally convenient for serial application in factoring if the quotient is written next to the regrouped dividend (as in Figures 16, 17, 18). The regrouped dividend can be written either below or above the original dividend.

LOW STRESS RESEARCH, A PRECIS

There are 20 low stress studies known to the author. Of these, four are the result of professional projects, seven are doctoral dissertations and nine are masters level theses or seminar papers. Most of the work was withheld from other publication in order to be published in its entirety, and with additional comment, as University of Maryland Arithmetic Center Monographs. Regrettably, that project was delayed, but may be completed in the near future. All of the studies are directed toward several questions related to low stress algorithms. However, they may be logically organized in seven categories and some immediate information from each category can be offered.

I. ADDITION X ACHIEVEMENT LEVEL

Both remedial intermediate students and preservice teachers show comparable significant gains in computational power when using low stress addition.

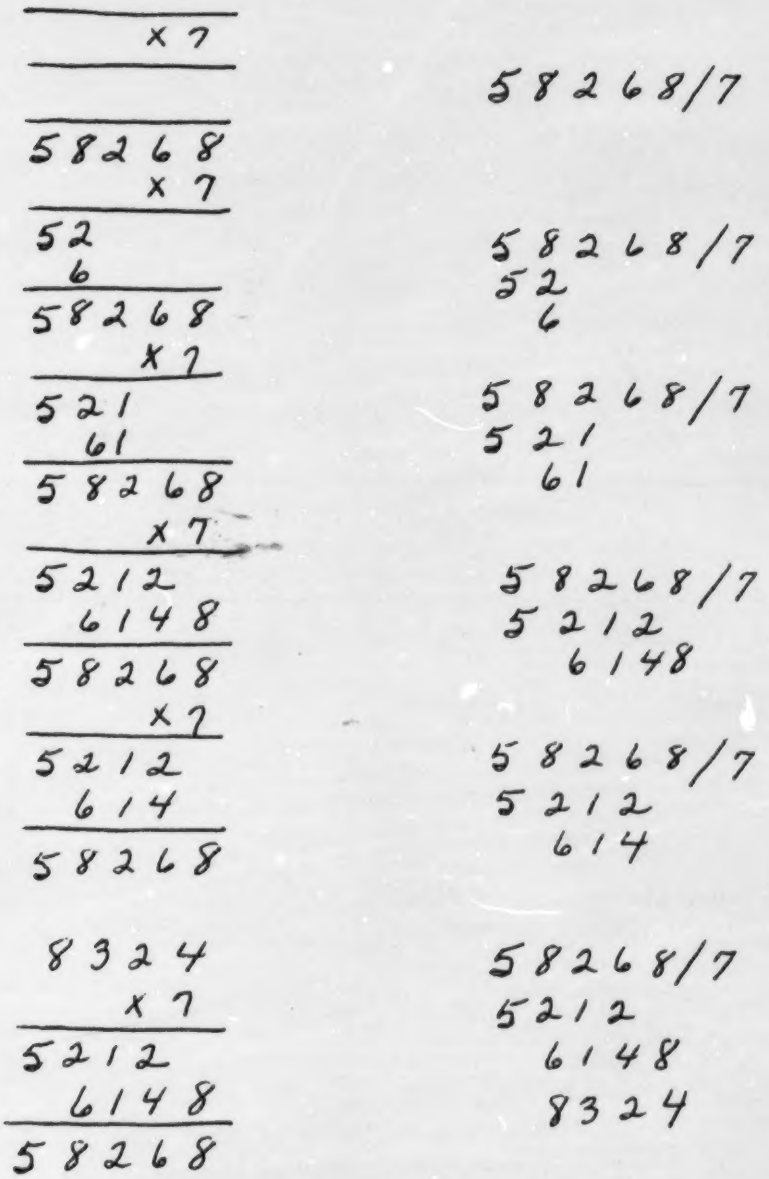


Figure 16

Low Stress Short Division
First Level and Reversed
Low Stress Multiplication

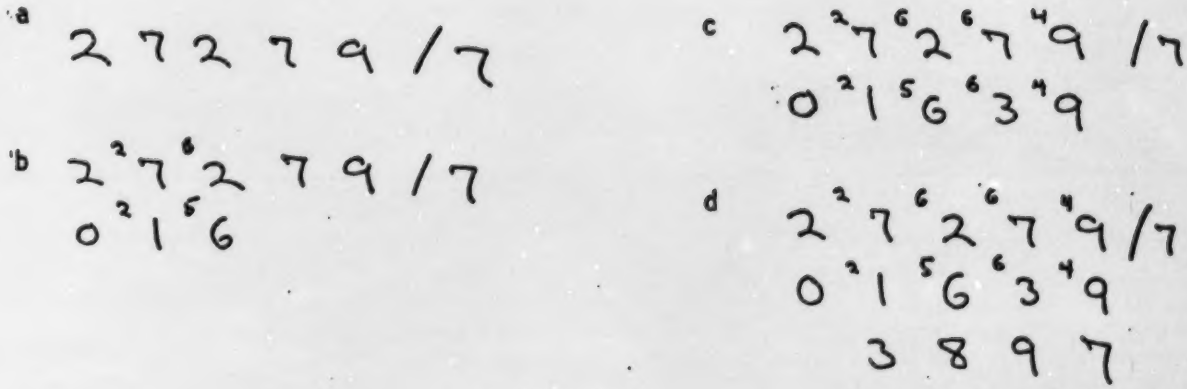


Figure 17

Low Stress Short Division
Second Level

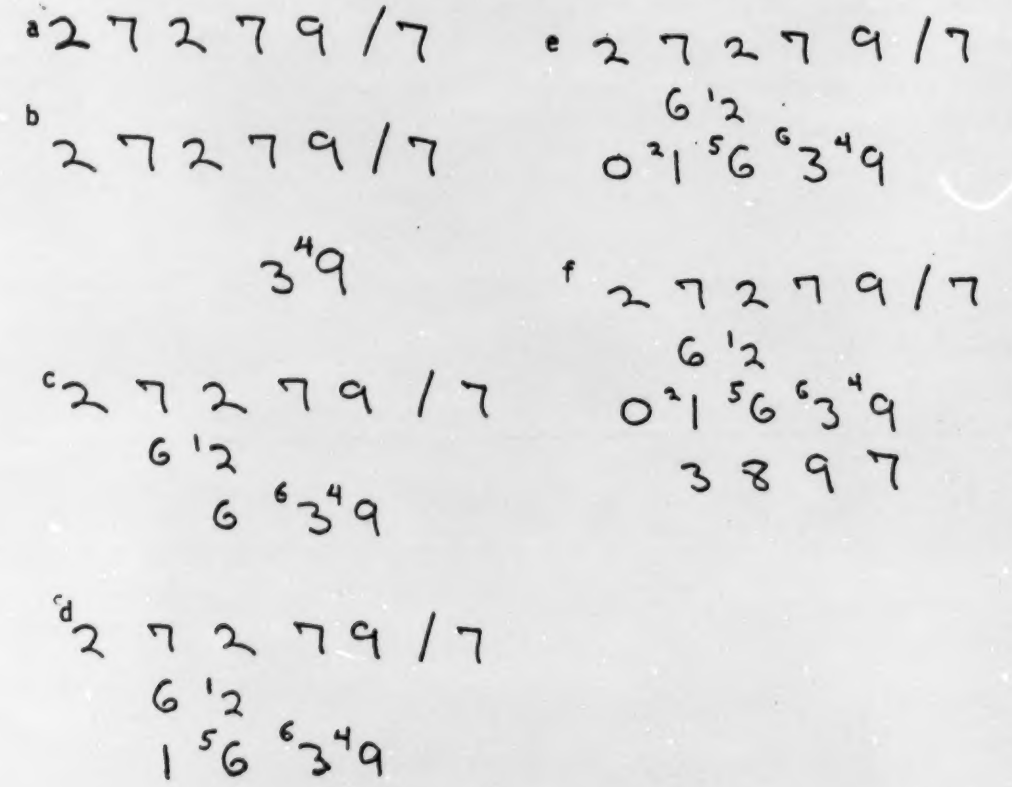


Figure 18

Low Stress Factoring Division

Low stress factoring division involves reversing the direction of regrouping so that it is the same as in subtraction.

Note that this procedure is simplified for the prime digits by the fact that knowledge of units place in a multiple of 3 or 7 determines tens place of that multiple (no repetitions in units place of the MOOF Column) and by the triviality of divisibility rules for 2 and 5.

Hutchings, L. and Cooke, N. "Rapid Development of a Basic Computational Skill: An Example." (1969)

Carlson, R. "The Relative Effectiveness of Hutchings' Full Record Algorithm for Addition and the Conventional Algorithm for Addition with Difficult Summation Problems in a Numerically Competent Adult Population." (1973)

II. ADDITION X ETHNIC BASE

Computational gains for low stress addition are significant, and its introduction procedures are relatively easy, in upper middle class groups. Low stress introduction procedures for disadvantaged inner city blacks may require the use of previously trained local black teachers as opposed to imported white workshop teachers. When the low stress program is used, inner city students show significant gains comparable to other populations.

Hutchings, B. "An Examination Across a Wide Range of Socioeconomic Circumstance, of a Format for Field Research of Experimental Numerical Computation Algorithms, an Instrument for Measuring Computational Power under any Concise Numerical Addition Algorithm and the Differential Effects of Short Term Instruction in Two Experimental Numerical Addition Algorithms and Equivalent Practice with the Conventional Algorithm." (1972)

Burton, T. "Examination of an Experimental Addition Algorithms Suitability for Inner City Sixth Grades When Presented in a Culturally Specific Format." (1972)

Perl, L. "Rapid Improvement of Addition Computation with Hutchings' Full Record Algorithm." (1972)

III. ADDITION X REINFORCEMENT

Significant gains in low stress additions, as compared to conventional addition, will occur in the absence of positive reinforcement or even informational feedback. Token economy reinforcement (candy) has significantly less effect than algorithm used when the low stress addition algorithms is included in the comparison. Disruptive anxiety generating moderation has significantly less effect on low stress algorithms than on conventional algorithms.

Phillips, P. "Examination of a Low Feedback Format of Instruction for an Experimental Addition Algorithm and an Addition Power Test with Regard to their Suitability for use with Fifth and Sixth Grades in an Open Classroom Situation." (1972)

Ashlock, R., and Hutchings, B. "A Comparison of the Low Stress Addition Algorithm for Performance Speed and Associated Accuracy at Different Levels of Problem Difficulty." (1973)

Alessi, G. "An Examination of the Effects of Using the Hutchings' 'Low Fatigue' Algorithm on Children's Addition Scores when Tested at Different Levels of Problem Difficulty and Under Varying Conditions of Token Economy Reinforcement." (1973)

Boyle, M. "Effects of Social Reinforcement under Simulated Test and Non-Test Conditions on the Computational Rate and Accuracy of Children Using Either the Conventional Algorithm or Hutchings' 'Low Stress" Algorithms." (1973)

IV. ADDITION FACTS
SUBTRACTION FACTS
ADDITION
SUBTRACTION
X ACHIEVEMENT LEVEL

As computational algorithms do not provide basic fact information, a method of removing basic fact deficiencies must be provided before the full advantages of improved algorithms can be realized by primary school and special education students. A low stress procedure called "zero point counting" and a number line of discrete counters (non-continuous intervals) appears to meet this need. Using this procedure and low stress algorithms, primary school students can show significant gains with respect to corresponding conventional methods in a short time and special education students show significant, especially dramatic gains. In fact, with the special education population, the differences are nearly dichotomous. That is, with low stress procedures, they acquire a normal competence and without these procedures, they do not.

Lester, L. "An Investigation into the Acceleration of Basic Quantitative Skills of Educationally Disadvantaged Rural Primary School Children Using Hutchings' Low Stress Procedures." (1978)

Dashiell, W. H. "An Analysis of Changes in Affect and Changes in Both Computational Power and Computational Stamina Occurring in Regular Elementary School Children After Instruction in Hutchings' 'Low-Stress' Addition Algorithm, Practice with Unusually Large Examples and Exposure to One of Two Alternate Performance Options." (1973)

Dashiell, W. H. "An Analysis of Changes in Affect and Changes in Both Computational Power and Computation Stamina Occurring in Mildly and Moderately Exceptional Elementary School Children After Instruction in Hutchings' 'Low-Stress' Addition Algorithm, Practice with Unusually Large Examples and Exposure to One of Two Alternate Performance Options." (1973)

Kalin, M. and Shami, M. "Accelerating the Learning Rate of Students with Mild Intellectual Limitations in Acquiring Basic Computational Skills." (1972)

V. SUBTRACTION X PRESENTATION

Intermediate level students show significant gains in subtraction using the low stress procedures when compared to controls using conventional procedures. These gains are approximately the same whether low stress subtraction is taught "meaningfully or mechanically." (Alessi suggests that Brownell's (1949) description of meaningful versus mechanical learning implies that there can be no non-meaningful presentation of low stress procedures because the low stress notation is intrinsically meaningful.)

Barrash, B. "The Relative Effects of Brief Instruction in the 'Low Stress' Subtraction Algorithm and Matched Practice in Conventional Subtraction." (1972)

Berger, B. "An Experimental Comparison of the Effects of Verbal Role and Verbal Explanatory Teaching Procedures on Computation with Hutchings' Low Stress Algorithm for Subtraction." (1977)

VI. TEST DEVELOPMENT, ADDITION AND SUBTRACTION

Experimental selection response forms of the incrementally difficult addition test appear not to merit development. A formula to describe the difficulty of any subtraction problem is attempted and several equivalent forms of a subtraction test developed.

Kurt, N. et al. "Process Variables of Subtraction. Beginning of a Multiple Regression Equation Predicting Accuracy of Response and Development of a Diagnostic Instrument." (1974)

Ying, N. "Parallel Measurements as a Function of Response Format; an Experimental Investigation of the Effects of Response Format on the Consistency of Variances, Convariances and Group Means." (1975)

VII. MULTIPLICATION X ACHIEVEMENT LEVEL

Significant gains in computational power occur with intermediate middle school students, at various achievement levels, when groups using low stress multiplication are compared with controls using conventional multiplication.

Leaf, M. "An Examination of the Low Stress Addition Algorithm, Basic Low Stress Multiplication Algorithm and Combined Low Stress Multiplication Algorithm in Intensely Sequenced Instruction Units in a Sixth Grade Self-contained Classroom." (1973)

Gordon, E. "An Examination of a Format for Field Research of Certain Experimental Numerical Computation Algorithms, an Instrument for Measuring Computational Power under any Concise Numerical Algorithm, and the Differential Effects of Short Term Instruction in

Experimental Algorithms and Equivalent Practice with the Conventional Addition and Multiplication Algorithms." (1973)

Berger, B. "A Comparison of Two Variations of the Low Stress Multiplication Algorithm and the Conventional Multiplication Algorithm." (1974)

All of the work above is, in the author's opinion, experimentally rigorous. However, it is all statistically conventional. That is, all of these studies employ the usual ANOVA, ANCOVA, and t procedures and use the medium to large groups required by those procedures. Consequently, they cannot offer statistical models to the very small group or single subject investigations which so commonly and urgently present themselves in clinical situations. The unfamiliarity of appropriate statistic and design combinations for such studies becomes a serious problem as the number of clinics is increased and the acceptability of subjective investigations is decreased. A possible solution and certainly an extremely interesting development, is the initiation of arithmetically directed (low stress) and clinically oriented, small group and single subject exploratory intervention investigations by Galen Alessi of the University of Western Michigan's Psychology Department. The techniques are experimentally, inferentially rigorous but highly compatible with the logistics of real life clinical situations. The author believes there are many interesting, important low stress clinical applications and inquiries that would be well served by the exploratory intervention approach. This approach can take advantage of the continuous diagnostic information provided by the complete record of binary events occurring in low stress notation.

However, whether clinics or classrooms obtain only a small number of the questions about low stress procedures have been asked. A tremendous amount of exciting research is waiting. Studies should focus on:

1. Why low stress procedures work
2. Extensions of the procedures e.g., to rational numbers
3. Modes of presentation x population

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and equivalent practice with the conventional addition and multiplication algorithms. University of Maryland Arithmetic Center, unpublished Master of Arts thesis, 1973.

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